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INTERMEDIATE TRIGONOMETRY

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U. N. DHUR & SONS, PRIVATE LTD.

BOOKSELLERS & PUBLISHERS

15. BANKIM CHATTERJEE STREET, CALCUTTA 12

Rupees 6.50 only.

Published by DWIJENDRANATH DHUR L.L.B. For U. N. DHUR & SONS PRIVATE LTD. 15, Bankim Chatterjee Street, Calcutta 12

22ND EDITION-1940

Printed by
TRIDIBESH BASU,
THE K. P. BASU PRINTING WORKS,
11. Mohendra Gossain Lane, Calcutta 6

PREFACE TO THE FIRST EDITION

THIS book, as its name indicates, is meant to be a text-book for the Intermediate students of Indian Universities, especially the University of Calcutta. Regarding the subject-matter, we have tried to make the exposition clear and concise, without going into unnecessary details. A good number of examples has been worked out by way of illustrations, and examples set have been carefully selected.

Important formulæ and results have been given at the beginning of the book for reference. Calcutta University questions of recent years are given at the end, to give the students an idea of the standard of the examination.

It is hoped that the book will meet the requirements of those for whom it is intended and we shall deem our labours amply rewarded if the students find the book useful to them.

The book had to be hurried through the press practically within the period of a fortnight, and we must thank the authorities and officers of the K. P. Basu Printing Works, Calcutta, who, in spite of their various preoccupations had the kindness to complete the printing in such a short period of time.

Any criticism, correction and suggestion towards improvement will be thankfully received.

B. C. D. B. N. M.

PREFACE TO THE FIFTH EDITION

THIS edition is practically a reprint of the fourth edition; only a new chapter dealing with harder problems on Heights and Distances, Summation of Finite Trigonometrical series, and Elimination has been added in the end to cover the syllabuses of some other Indian Universities.

B. C. D.

B. N. M.

PREFACE TO THE TWENTY-SECOND EDITION

ALTHOUGH this edition is practically a reprint of the previous edition, it contains all those topics that are mentioned in the revised syllabus of Trigonometry of the Calcutta University.

B. C. D.

B. N. M.

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GREEK LETTERS USED IN THE BOOK

α	(Alpha)	0	(Thetā)
β	(Bētā)	π	(Pai)
γ	(Gamma)	φ	(Phai)
δ	(Delt $\overline{\mathrm{a}}$)	ψ	(Psi)
	⊿ (Deltā)		

Note. The notation C. U. used at the end of any example means that the example was set in Intermediate Examination of the Calcutta University.

TRIGONOMETRY SYLLABUS FOR

I. A. & I. Sc. EXAMINATIONS

Measurement of angles. Trigonometric ratios and their graphs. Elementary Trigonometrical formulæ. Trigonometric ratios of associated angles. Summation Theorems. Transformation of products and sums. Multiple and submultiple angles. Trigonometric Equations and General Values. Inverse Circular Functions. Relations between sides and angles of a triangle. In-radius, Circum-radius and Area of a triangle. Practical solutions using log tables. Simple problems in heights and distances.

TRIGONOMETRY SYLLABUS FOR PRE-UNIVERSITY AND ENTRANCE COURSES

Measurement of angles—Sexagesimal and circular measures. Definition of trigonometrical ratios, their mutual relations. Deductions of the values of the trigonometrical ratios of 0°, 30°, 45°, 60°, 90°. Trigonometrical ratios of associated angles. Addition and subtraction formulæ. Transformation of products and sums of trigonometrical ratios. Multiple and sub-multiple angles (simple cases). General values, Solution of trigonometrical equations. Inverse circular functions. Trigonometrical identities. Relation between sides and angles of a triangle, area, inradius and circum-radius of a triangle. Solution of triangles with use of log tables. Graphs of simple trigonometrical functions. Simple problems of heights and distances.

TRIGONOMETRY SYLLABUS FOR HIGHER SECONDARY COURSE

Class IX

Measurement of angles in degrees, minutes, seconds and in radians. Definition of trigonometrical ratios of an acute angle. Trigonometrical ratios of the standard angles—0°, 30°, 45°, 60°, 90°, (undefined values such as tan 90°, cot 0°, to be excluded). Simple identities connecting the ratios of an angle immediately derivable from a right-angled triangle. Trigonometrical ratios of complementary angles.

Easy problems on heights and distances reducible to the solution of right-angled triangles involving the standard angles above.

Class X

Trigonometrical ratios of any angle: Trigonometrical ratios of angles associated with a given angle; Addition and subtraction formulæ; Transformation of products and sums; Multiple and sub-multiple angles.

Class XI

Graphs of simple trigonometric functions.

Trigonometric equations and general values; Inverse Circular Functions.

Relation between sides and angles of a triangle; Inradius, circum-radius and area of a triangle; Practical solution of a triangle with the help of logarithms; Simple problems of heights and distances.

IMPORTANT FORMULÆ AND RESULTS

1 degree = '01745 radians nearly.

2 right angles = 180° = π radians.

 $\pi = \frac{32}{7} = 3.1416$ approximately.

Radian measure of an angle at the centre of a circle

II.
$$\sin^2\theta + \cos^2\theta = 1$$
; $\sec^2\theta = 1 + \tan^2\theta$; $\cos^2\theta = 1 + \cot^2\theta$.

III. $\sin^2\theta + \cos^2\theta = 1 + \cot^2\theta$.

III. $\sin^2\theta = 0$; $\cos^2\theta = 1 + \cot^2\theta$.

III. $\sin^2\theta = 0$; $\cos^2\theta = 1$; $\tan^2\theta = 0$.

 $\sin^2\theta = 0$; $\cos^2\theta = 1$; $\tan^2\theta = 0$.

 $\sin^2\theta = 0$; $\cos^2\theta = 1$; $\tan^2\theta = 0$.

 $\sin^2\theta = 0$; $\cos^2\theta = 1$; $\tan^2\theta = 0$.

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 $\sin^2\theta = 1$; $\cos^2\theta = 0$.

 $\sin^2\theta = 0$; $\cos^2\theta = 1$; $\tan^2\theta = 0$.

 $\sin^2\theta = 0$; $\cos^2\theta = 0$; $\tan^2\theta = 0$.

IV.
$$\sin(-\theta) = -\sin\theta$$
; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.
 $\sin(90^{\circ} - \theta) = \cos\theta$; $\sin(90^{\circ} + \theta) = \cos\theta$.
 $\cos(90^{\circ} - \theta) = \sin\theta$; $\cos(90^{\circ} + \theta) = -\sin\theta$.
 $\tan(90^{\circ} - \theta) = \cot\theta$; $\tan(90^{\circ} + \theta) = -\cot\theta$.
 $\sin(180^{\circ} - \theta) = \sin\theta$; $\sin(180^{\circ} + \theta) = -\sin\theta$.
 $\cos(180^{\circ} - \theta) = -\cos\theta$; $\cos(180^{\circ} + \theta) = -\cos\theta$.
 $\tan(180^{\circ} - \theta) = -\tan\theta$; $\tan(180^{\circ} + \theta) = \tan\theta$.

(positive)

V.
$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

 $\sin (A - B) = \sin A \cos B - \cos A \sin B$
 $\cos (A + B) = \cos A \cos B - \sin A \sin B$
 $\cos (A - B) = \cos A \cos B + \sin A \sin B$
 $\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
 $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
 $\tan (A + B + C)$

(positive)

 $= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$

VI.
$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

 $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
 $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
 $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.
VII. $\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$

$$\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}.$$
VIII. $\sin 2A = 2 \sin A \cos A$
 $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$
 $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$
 $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}; \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$
 $4 = 2 \sin^2 A$
 $1 + \cos 2A = 2 \cos^2 A$
IX. $\sin 3A = 3 \sin A - 4 \sin^3 A$
 $\cos 3A = 4 \cos^3 A - 3 \cos A$
 $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$
X. $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$
 $\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = 2 \cos^2 \frac{\theta}{2} - 1 = 1 - 2 \sin^3 \frac{\theta}{2}$
 $1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$
 $1 - \cos \theta = 2 \cos^2 \frac{\theta}{2}$
 $1 - \cos \theta = 2 \cos^2 \frac{\theta}{2}$
 $1 - \cos \theta = \tan^2 \frac{\theta}{2}.$

XI. If
$$\sin \theta = \sin \alpha$$
, then $\theta = n\pi + (-1)^n \alpha$.
If $\cos \theta = \cos \alpha$, then $\theta = 2n\pi \pm \alpha$.
If $\tan \theta = \tan \alpha$, then $\theta = n\pi + \alpha$.
If $\sin \theta = 0$, or, $\tan \theta = 0$, $\theta = n\pi$.
If $\cos \theta = 0$, or, $\cot \theta = 0$, $\theta = (2n+1)^{\frac{\pi}{2}}$.
If $\sin \theta = 1$, $\theta = (4m+1)^{\frac{\pi}{2}}$; if $\sin \theta = -1$, $\theta = (4m-1)^{\frac{\pi}{2}}$.
If $\cos \theta = 1$, $\theta = 2m\pi$; if $\cos \theta = -1$, $\theta = (2m+1)\pi$.

XII.
$$\sin^{-1}x + \cos^{-1}x = \frac{1}{2}\pi$$

 $\tan^{-1}x + \cot^{-1}x = \frac{1}{2}\pi$
 $\sec^{-1}x + \csc^{-1}x = \frac{1}{2}\pi$
 $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$
 $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

 $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{x + y + z - xyz}{1 - yz - zx - xy}$ $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1 - y^2} \pm y\sqrt{1 - x^2}\right\}$ $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left\{xy \mp \sqrt{1 - x^2}, \sqrt{1 - y^2}\right\}.$

XIII. $\log_a mn = \log_a m + \log_a n$ $\log_a \frac{m}{n} = \log_a m - \log_a n$; $\log_a m^n = n \log_a m$; $\log_a m = \log_b m \times \log_a b$; $\log_a 1 = 0$; $\log_a a = 1$.

XIV.
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

 $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
 $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$; \rangle
 $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$$a = b \cos C + c \cos B.$$

$$b = c \cos A + a \cos C.$$

$$c = a \cos B + b \cos A.$$

$$A = \sqrt{(s-b)(s-c)}$$

$$bc$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2A}{bc}$$

$$\sin B = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2A}{ca}$$

$$\sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2A}{ab}$$

$$\triangle = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{3}ab \sin C$$

$$= \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } 2s = a+b+c$$

$$= \frac{abc}{4k}.$$

$$R = \frac{a}{2} \sin A = \frac{b}{2} \sin B = \frac{abc}{2} \sin C = \frac{abc}{4A}.$$

$$r = \frac{A}{s} = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$$

$$= (s-a) \tan \frac{1}{2}A = (s-b) \tan \frac{1}{2}B = (s-c) \tan \frac{1}{2}C.$$

$$r_1 = \frac{A}{s-a} = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$$

$$= s \tan \frac{1}{2}A.$$

$$r_2 = \frac{A}{s-b} = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C$$

$$= s \tan \frac{1}{2}B.$$

$$r_3 = \frac{A}{s-c} = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C$$

$$= s \tan \frac{1}{2}C.$$

IMPORTANT RESULTS

- 1. If $A+B+C=\pi$, then
 - (i) $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.
 - (ii) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.
 - (iii) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
 - (iv) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
 - (v) $\cos 2A + \cos 2B + \cos 2C$ = $-4 \cos A \cos B \cos C - 1$.
 - (vi) $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$.
 - (vii) $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.

(viii)
$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}$$

= $1 + 4 \sin \frac{B + C}{4} \sin \frac{C + A}{4} \sin \frac{A + B}{4}$.

(ix)
$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}$$

= $4 \cos \frac{B+C}{4} \cos \frac{C+A}{4} \cos \frac{A+B}{4}$.

(x)
$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1$$
.

(xi)
$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

2.
$$Lt \underset{\theta \to 0}{\sin \theta} = 1$$
; $Lt \underset{\theta \to 0}{\cot \theta} = 1$; $Lt \underset{\theta \to 0}{\tan \theta} = 1$.

3. Area of a circle of radius $r = \pi r^2$.

Perimeter of a circle of radius $r = 2\pi r$.

INTERMEDIATE TRIGONOMETRY

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CHAPTER I

MEASUREMENT OF ANGLES

1. TRIGONOMETRY, as indicated by its very name, originally meant a subject which dealt with the methods of measurement of triangles. At present its scope has widened, and now it means a subject which deals with the measurements relating to any angle, not necessarily an angle of a triangle.

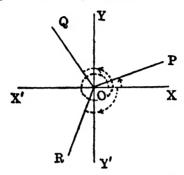
2. Angles in Trigonometry.

In Geometry, angles are supposed to be formed by the intersection of two straight lines and are always restricted to lie between 0° and 360°, being acute, obtuse or reflex. Moreover, they are always positive, negative angles having no meaning. In Trigonometry however, the idea of an angle is much more general.

An angle in Trigonometry is supposed to be formed by the revolution of a straight line which starts from an initial position coinciding with one arm, and traces out the angle by its revolution about one extremity until it reaches the final position coinciding with the other arm.

For instance, the angle XOP is formed by the revolution of a line which starts from the initial position OX, and revolving in the anti-clockwise direction, traces out the angle XOP which is acute. The same line again, starting from OX and revolving in the anti-clockwise direction may make a complete revolution and further move up to the position OQ. The angle formed in this case is more than

five right angles. Now revolutions may be clockwise or anti-clockwise. It is conventional to consider angles formed by the anti-clockwise revolution of the revolving line to be positive. Angles formed by clockwise revolutions of the



revolving line will then be considered negative angles. For example, the angle XOR measured in the clockwise_direction from the initial position OX is a negative angle.

Thus, angles in Trigonometry may be of any magnitude and may be positive as well as negative.

OX being the initial position of the revolving line, produce XO to X', and let YOY' be the perpendicular line. The whole plane is thus divided into four quadrants, the first being XOY, the second YOX', the third X'OY' and the fourth Y'OX. If we contemplate an angle say $+920^{\circ}$ to be traced out by the revolving line, the line must have completed two complete revolutions, thereby describing $2 \times 360^{\circ} = 720^{\circ}$, and have further traced out an angle 200° , so that the final position of the revolving line is in the third quadrant. Similarly, if we consider an angle -1354° , the final position of the revolving line is in the first quadrant, for $-1354^{\circ} = -360^{\circ} \times 3 - 274^{\circ}$.

It should be noted that if two angles differ by complete multiples of 360°, the starting line being the same, the final position of the revolving line will be coincident for the two angles. For example, the angles 255° and -105° will have the final positions of the revolving line same, if both start from the same initial position.

3. Units of measurement of angles.

We should now define the different systems of units used for the measurement of angles. In defining a unit however, a standard angle, which has no reference to any particular system of unit, should form the basis, and such a standard angle is a right angle. A right angle is defined in books on Geometry to be an angle which any straight line standing on another makes with it, when the two adjacent angles formed are equal to one another. A right angle is always the same everywhere, and it thus forms a suitable basis to start with, in defining the different systems of measurement of angles.

There are three systems of units used in Trigonometry for measurement of angles, viz.,

- (i) Sexagesimal unit.
- (ii) Centesimal unit.
- (iii) Circular unit.

Sexagesimal* System. In this system, a right angle is divided into 90 equal parts, each being called a degree. A degree is again divided into 60 sexagesimal minutes, and each minute is further subdivided into 60 sexagesimal seconds, so that

- 1 rt. angle=90° (degrees)
- 1° = 60' (sexagesimal minutes)
- 1' = 60" (sexagesimal seconds)

^{*}So called, since the subdivisions are mostly by sixtleth parts. It is also called the Common or the English System.

Centesimal† System. In this system, the subdivisions of a right angle are as follows:

1 rt. angle=100^{\$} (grades)

1² = 100' (centesimal minutes)

1' = 100" (centesimal seconds).

Note. It may be noted that 1' (centesimal minute) is not the same as 1' (sexagesimal minute), the former being $\frac{1}{100 \times 100}$ of a right angle and the latter being $\frac{1}{90 \times 60}$ of a right angle, so that the first is $\frac{27}{100}$ th part of the second. Similarly, 1'' is less than 1", being only $\frac{2}{120}$ th part of it.

The connection between the two systems of units may be effected through a right angle, remembering that 1 right angle $= 90^{\circ} = 100^{\circ}$, so that $9^{\circ} = 10^{\circ}$. Any angle in the first system may be reduced to degrees, and then multiplied by $\frac{10^{\circ}}{2}$ will be reduced to grades. Similarly, an angle in the second system may be changed to the first.

We shall presently deal with the third system, namely the circular system.

4. Theorem. In all circles, the circumference bears a constant ratio to its diameter.

Take any two circles of any radii, and place them with a common centre O. In one, let ABCD... be an inscribed regular polygon of n sides. Let A', B', C',... be the points of intersection of the radii OA, OB, OC,... with the other circle. It is easily seen that A'B'C'... is also a regular polygon of n sides, inscribed in the second circle. Now OA = OB as also OA' = OB', so that in the triangles OAB, OA'B',

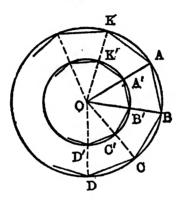
[†] So called because the subdivisions are by hundredths. It is also called the *French System*.

MEASUREMENT OF ANGLES

OA: OA' = OB: OB', and angle O is common. The two triangles are therefore similar. Hence, AB: A'B' = OA: OA'.

Thus.

perimeter of polygon ABCD... = n.AB = OAperimeter of polygon A'B'C'D'... = n.A'B' = OA'



This being true, whatever the number of sides n may be, making n infinitely large, the perimeters of the polygons can be made practically coincident with the circumferences of the corresponding circles, and thus we deduce that

circumference of the circle
$$ABCD...$$
circumference of the circle $A'B'C'D'...$

a., = radius of circle $ABC...$
radius of circle $A'B'C'...$

Thus circumference of any circle: its radius is the same for all circles. As diameter is twice the radius, we deduce that the circumference of any circle bears a constant ratio to its diameter.

This constant ratio is denoted by the Greek letter π . Its actual value has been determined by methods which are outside the scope of the present book, by some mathematicians

to more than 500 places of decimals. An approximate value commonly used is $\frac{3}{4}$. A more accurate value is $\frac{3}{1}$.

Expressed in decimal, the value is nearly 3'14159...

Hence, if r be the radius of a circle, d its diameter,

the circumference $= \pi d = 2\pi r$.

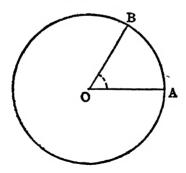
where $\pi = 3.14159 \dots = \frac{21}{7}$ roughly.

5. Circular Unit or Radian Measure.

In any circle, if we take an arc whose length is equal to the radius of the circle, the angle which this arc subtends at the centre, is called a *radian*, and is written as 1°.

We shall now show that with reference to whichever circle it may be defined, a radian is a constant angle, and hence it may be used as a suitable unit for measurement of angles, which is known as the circular unit.

Theorem I. A radian is a constant angle.



Let AB be an arc of any circle with centre O, whose length is equal to its radius OA. By definition, $\angle AOB = 1$ radian. Since angles at the centre of a circle are proportional to the arcs which subtend them, and the whole angle

round O subtended by the complete circumference being known from Geometry to be 4 right angles, we get

$$\frac{\angle AOB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{whole circumference}} = \frac{\text{radius}}{\text{circumference}}$$

$$i.e., \quad \frac{1 \text{ radian}}{4 \text{ rt. } \angle s} = \frac{r}{2\pi r} = \frac{1}{2\pi}, \quad r \text{ being the radius.}$$
Hence, $1 \text{ radian} = \frac{2}{r} \text{ rt. angle.}$

a radian is a constant angle (π being constant).

Note. We thus see that whatever be the radius of the circle with reference to which a radian is defined, its magnitude is the same.

From above, π radians=180°.

... 1 radian =
$$\frac{180}{\pi} = \frac{180}{3.14159} = 57.29577$$
 degrees = 57° 17′ 44.8″ nearly,

... 1 degree = '0174533 radians nearly.

In higher mathematics so far as theoretical investigations are concerned, as a matter of convenience, angles are usually measured in the circular unit, i.e., in radians. In this connection we may state the following theorem:

Theorem II. The measure of any angle in radians is expressed by the ratio of the arc of any circle subtending that angle at its centre, to the radius.

Let XOP be any angle.

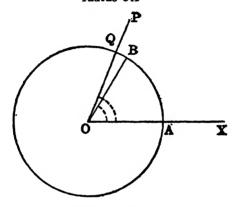
With centre O and any radius OA draw a circle, and let AQ be the arc which subtends the angle XOP at the centre O. Let AB be the arc whose length is equal to the radius AO, so that, by definition, $\angle AOB$ is one radian.

Now from Geometry, angles at the centre of a circle are proportional to the arcs which subtend them.

TRIGONOMETRY

Hence,
$$\angle XOP = \frac{\text{arc } AQ}{\text{arc } AB} = \frac{\text{arc } AQ}{\text{radius } OA}$$
, or, $\angle XOP = \frac{\text{arc } AQ}{\text{radius } OA}$, radius OA

i.e., $\angle XOP = \frac{\text{arc } AQ}{\text{radius } OA}$ of a radian.



Thus, if θ be the radian-measure of the $\angle XOP$, s be the length of the arc AQ, and r the radius of the circle, then

$$\theta = \frac{8}{r}$$
 or, $s = r\theta$.

Note. In higher mathematics, when an angle is expressed in radian-measure, the unit is generally implied and not expressed, so that, when the measure of an angle is given without the unit being mentioned, we should always understand it to be in radians. For example, 'an angle is $\frac{\pi}{2}$ means that the angle is $\frac{\pi}{2}$ radians, which converted to degrees is 90° i.e., one right angle.

6. In working out examples, relations between the three systems of units should be carefully remembered, namely

1 rt.
$$\angle -90^{\circ} - 100^{g} - \frac{\pi}{2}$$
 radians, whence, $\pi^{\circ} - 180^{\circ}$.

Ex. 1. Express

(i) 63° 22′ 40'8" in centesimal measure

and (ii) 203° 58' 73" in radians.

Here, (i)
$$63^{\circ} 22' 40'8'' = 63\frac{180}{500} \text{ deg.} = \frac{31}{500} \frac{632}{500} \times \frac{1}{90} \text{ rt.} \angle$$

= $\frac{31}{500} \frac{632}{500} \times \frac{1}{90} \times 100 \text{ grades} = \frac{3531}{500} \text{ grades}$
= $70^{\circ} 42'$.

(ii)
$$203^g$$
 58' 73" = 203 5873 grades
= 2.035873 rt. $\angle = 2.035873 \times \frac{\pi}{2}$ radians
= 1.0179365π radians.

Ex. 2. Two angles of a triangle are 72° 53′ 51″, and 41° 22′ 50″ respectively. Find the third angle in radians.

41° 22' 50" = 41'2250 grades
=
$$\frac{41'225 \times 9}{10}$$
 degrees [9° = 10°]
= 37'1025 degrees
= 37° 6' 9".

The sum of the two given angles is therefore $72^{\circ} 53' 51'' + 37^{\circ} 6' 9'' = 110^{\circ}$.

The sum of the three angles of a triangle being 180°, the third angle is

$$180^{\circ} - 110^{\circ} = 70^{\circ} = 70^{\circ} \times \frac{\pi}{180}$$
 radians [$\pi^{\circ} = 180^{\circ}$]
= $\frac{7\pi}{18}$ radians.

Ex. 3. Divide $\frac{\pi}{4}$ radians into two parts such that the number of sexagesimal minutes in one may be to the number of centesimal seconds in the other part as 27:2500.

We have
$$\frac{\pi}{4}$$
 radians = $\frac{\pi}{4} \times \frac{2}{\pi}$ rt. $\angle = \frac{1}{2}$ rt. \angle .

Let x be the number of centesimal seconds in the second part, so that $\frac{127}{10}x$ is the number of sexagesimal minutes in the first part.

Now,
$$x'' = \frac{x}{100 \times 100 \times 100} \text{ rt. } \angle$$

and $\frac{27}{2500}x' = \frac{27x}{2500 \times 60 \times 90} \text{ rt. } \angle = \frac{x}{500000} \text{ rt. } \angle$,
 $\therefore \frac{x}{1000000} + \frac{x}{500000} = \frac{1}{2}$,
whence $x = \frac{500000}{3}$.

Thus, second part is
$$\frac{500000}{3}$$
 = $\frac{500000}{3 \times 100 \times 100 \times 100}$ rt. \angle = $\frac{1}{6}$ rt. \angle = 15°, and as the sum of the two parts is $\frac{1}{2}$ rt. \angle i.e., 45°, the first part is 30°.

The two parts are therefore 30° and 15°.

Ex. 4. The angles of a quadrilateral are in A.P., and the number of grades in the least angle is to the number of radians in the greatest as $100:\pi$. Find the angles in degrees.

Let the angles, expressed in degrees, be a, $a+\beta$, $a+2\beta$ and $a+3\beta$ respectively. Then

$$a + a + \beta + a + 2\beta + a + 3\beta = 360,$$
 ... (1)
i.e., $2\alpha + 3\beta = 180.$

Again the least angle, $a^{\circ} = \frac{10}{9}a^{0}$

and the greatest angle, $(a+3\beta)^{\circ} = (a+3\beta) \frac{\pi^{\circ}}{180}$

and so from the given condition,

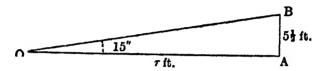
$$\frac{10}{9} \alpha/(\alpha+3\beta) \frac{\pi}{180} = 100/\pi,$$

or,
$$\frac{2a}{a+3\beta}=1$$
, whence $a=3\beta$.

:. using (1),
$$3a = 180$$
, or, $a = 60$ and $\beta = \frac{a}{3} = 20$.

Thus the angles are

Ex. 5. At what distance does a man, 5\frac{1}{2} ft. in height subtend an angle of 15"?



AB being the man subtending an angle 15" at O_1 let OA be r ft.

As the angle AOB is very small, so that AB is very small compared to AO, we may assume the small length AB to be practically a small arc of a circle whose centre is O. Now the measure of an angle in radians is the ratio of the arc which subtends it at the centre to the radius.

[See Art. 5]

$$\frac{15}{60 \times 60} \times \frac{\pi}{180} = \frac{5\frac{1}{2}}{r},$$
or, $r = \frac{11}{2} \times \frac{180 \times 60 \times 60}{15 \times \pi}$ ft.
$$= \frac{11}{2} \times \frac{180 \times 60 \times 60 \times 7}{15 \times 22} \times \frac{1}{3 \times 1760} \text{ miles approx.}$$
= 14'32 miles nearly.

Examples I

1. Indicate the final position of a revolving line which has traced out the angle

(i)
$$1122^{\circ}$$
; (ii) $-810^{\circ} 29'$; (iii) $-617^{g} 51' 5''$ (iv) $\frac{18\pi}{5}$ radians.

- Express (i) 55° 12′ 36″ in centesimal measure;
 (ii) 195″ 35′ 24″ in degrees, minutes, and secs.
- **8.** How many radians are there in (i) 50^{o} 75' 50"; (ii) 18° 33' 45''?
- 4. Express in each system of angular measurement the angle between the minute-hand and the hour-hand of a clock at quarter to twelve.
- 5. If x^{ρ} be taken as the unit angle, and the angles 600° and 16° expressed in that unit be α and β respectively, find the relation between α and β .
- 6. The difference of two angles is 1°; the circular measure of their sum is 1; find the circular measure of the smaller angle.
- 7. Two angles are in the ratio 2:3, and the difference of their measure in grades and in degrees respectively is 2; find the angles in degrees.
- 8. An angle is the excess of $D^{\circ} M'$ over $G^{\circ} m'$. Find the ratio of this angle to a right angle.
- 9. The circular measure of a certain angle is equal to the ratio of the number of degrees in it to the number of centesimal minutes; find the magnitude of the angle in degrees.
- 10. With two units of angular measurements differing by 10°, the measure of an angle are as 3:2; determine the units.
- 11. If an angle standing upon an arc of length 'l' at the centre of a circle of radius 'r' be taken as unit, and three angles D° , G^{g} , and C circular units expressed in that unit be x, y, z respectively, show that

$$x: y: z = \frac{D\pi}{18}: \frac{G\pi}{20}: 100.$$

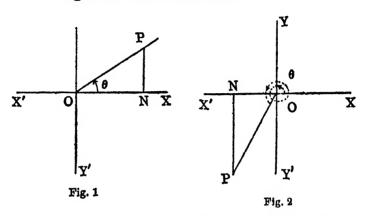
- 12. Three angles are in G.P. The number of grades in the greatest angle is to the number of circular units in the least as 800 to π , and the sum of the three angles is 126°. Find the angles in grades.
- 13. Divide 54° in three parts, such that the circular measure of the first exceeds that of the second by $\frac{\pi}{10}$, and the sum of the second and third is 30 grades.
- 14. Find at what times between 7 and 8 o'clock the angle between the two hands of a clock is (i) 60°, (ii) 155°.
- 15. The angles of a triangle are in A.P., and the number of radians in the greatest is to the number of grades in the least as π : 40. Find the angles in degrees.
- 16. In each of two triangles the angles are in G.P.; the least angle of one of them is three times the least angle in the other, and the sum of the greatest angles is 240°. Find the circular measure of the angles.
- 17. One angle of a quadrilateral is $\frac{3}{4}$ of another and the two other angles are $66\frac{2}{3}$ grades and $\frac{3}{4}\pi$ radians. Express the angles in degrees.
- 18. The angles of a polygon (which has no reflex angle) are in A.P. The least angle is $\frac{2}{3}\pi$ radians and the common difference is 5°. Find the number of sides.
- 19. The number of sides of two regular polygons are as m:n and the number of degrees in an angle of the first is to the number of grades in an angle of the second as p:q. Determine the number of sides in each polygon.
- 20. An arc of 50° in one circle equals one of 60° in another; find the radian-measure of an angle subtended at the centre of the first circle by an arc equal to the radius of the second.

- 21. Two regular figures are such that the number of degrees in an angle of one is to the number of degrees in an angle of the other as the number of sides in the first is to the number of sides in the second. The sum of the number of sides of the two figures being 9, determine the number of sides of each.
- 22. The wheel of a railway carriage is 4 ft. in diameter and makes 6 revolutions in a second; how fast is the train going?
- 23. The earth revolves round the sun in a circular orbit of radius 92700000 miles once a year. Find its velocity in miles per hour. If the apparent angular diameter of the sun observed from the earth be 32', find also the linear radius of the sun.
- 24. A tower subtends an angle of 10' when the observer is at a distance of 6 miles; find its height.
- 25. Find the radius of the earth, if an angle of 1° is subtended at its centre by an arc joining two places on it distant 69'1 miles.
- 26. A horse is tied to a post by a rope 27 feet long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far the horse will have gone when the rope has traced out an angle of 70°. $(n = \frac{2}{3})^2$
- 27. A man running along a circular track at the rate of 10 miles per hour, traverses in 36 seconds, an arc which subtends 56° at the centre. Find the diameter of the circle. $(\pi = \frac{2\pi}{3})$
- 28. An arc of 30° in one circle is double an arc in a second circle, the radius of which is three times the radius of the first. Show that the arc of the second circle subtends 5° at its centre.

CHAPTER 11

TRIGONOMETRICAL RATIOS*

7. Trigonometrical ratios defined.



Let θ be the measure of an angle XOP which may be supposed to be traced out by a revolving line starting from the initial position OX. From any point P on its other arm, draw a perpendicular PN on OX (produced if necessary, as in the second figure). A right-angled triangle is thereby formed. The trigonometrical ratios of the angle θ are defined as follows:—

Sine of the angle
$$\theta$$
, written as $\sin \theta = \frac{PN}{OP}$

i.e., $\frac{opposite\ side}{hypotenuse}$

Cosine of θ , written as $\cos \theta = \frac{ON}{OP}$

i.e., $\frac{adjacent\ side}{hypotenuse}$

^{*} For alternative definitions, see Appendix.

Tangent of
$$\theta$$
, written as $\tan \theta = \frac{PN}{ON}$

Cosecant of
$$\theta$$
, written as cosec $\theta = \frac{OP}{PN}$

Secant of
$$\theta$$
, written as $\sec \theta = \frac{OP}{ON}$

Cotangent of
$$\theta$$
, written as $\cot \theta = \frac{ON}{PN}$

In addition to these, we define two less important ratios of the angle θ which are sometimes used, as follows:—

Versed sine of angle θ , written as vers $\theta = 1 - \cos \theta$.

Coversed sine of angle θ , written as covers $\theta = 1 - \sin \theta$.

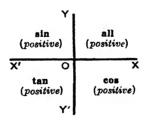
8. Signs of Trigonometrical ratios.

XOP being any angle, traced out by a revolving line which starts from OX, it has already been mentioned in the previous Chapter that the plane may by divided into four quadrants by the two perpendicular lines XOX' and YOY', the first being horizontal and the second vertical.

It is conventional, as in graphs, to consider distances measured along OX and OY as positive, and along OX' and OY' as negative. The distance measured along OP, the final position of the revolving line corresponding to the angle XOP, in whichever quadrant it may lie, is however always considered positive.

With this convention, if OP lies in the first quadrant as in Fig. (i) of the last article, the sides PN, ON and OP of the right-angled triangle OPN are all positive. Hence, all the trigonometrical ratios are positive. If OP lies in the third quadrant as in Fig. (ii), ON and PN are both negative, but OP is positive. Hence, from the definition of the trigonometrical ratios, $\sin XOP \left(-\frac{PN}{OP} \right)$ is negative, $\cos XOP \left(-\frac{ON}{OP} \right)$ is negative, $\tan XOP \left(-\frac{PN}{ON} \right)$ negative quantity is positive etc.

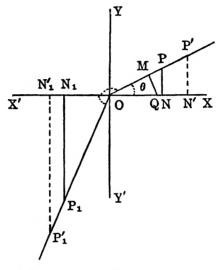
In this way, according to the final position of the revolving line (starting position being OX), we can determine the signs of the trigonometrical ratios of the angle XOP whether this angle traced out is positive or negative. If OP is in the first quadrant, the ratios are all positive. If OP falls in the second quadrant, sine and cosecant (which is evidently the reciprocal of sine), are positive; all the other ratios are negative. If OP be in the third quadrant, tangent and cotangent (which are reciprocals to each other) are positive; all the others are negative. In the fourth quadrant, cosine and secant are positive, others are negative. A symbolical figure will help the memory in this case, namely that according to the position of OP,



The positiveness of sine, cosine and tangent also implies the positiveness of their reciprocals namely, cosecant, secant and cotangent respectively.

9. Constancy of Trigonometrical ratios.

So long as an angle remains the same, its trigonometrical ratios are unique.



Let $XOP (= \theta)$ be any angle, and let PN and P'N' be drawn perpendiculars upon OX from any two points P and P' on OP. The two right-angled triangles OPN and OP'N' are similar. Hence, $\sin \theta$, whether we take it as $\frac{PN}{OP}$ or $\frac{P'N'}{OP'}$ is the same. If the angle be XOP_1 , when OP_1 is not in the first quadrant, the right-angled triangles P_1N_1O and $P'_1N'_1O$ are not only similar but also have their corresponding sides of the same sign. Hence, the trigonometrical ratios of the angle XOP_1 , whether defined from the triangle P_1N_1O or from $P'_1N'_1O$ are the same in magnitude as well as in sign. Thus for any given angle, the trigonometrical ratios are unique.

Note. In case of a positive acute angle like XOP, we might take any point Q on OX as well, and draw QM perpendicular upon OP, and define $\sin XOP$ to be opposite side i.e., QM, OQ, $\cos XOP$ to be OM etc. Now the two triangles QOM and PON are easily seen to be similar and both have their sides all positive; so that QM = PN, OM = ON or etc. Hence the trigonometrical ratios of the angle XOP, even if defined from triangle QOM, will have the same values.

It may also be noted that for angles of any magnitude, positive or negative, any of the two arms may be supposed to be coincident with OX, and then the magnitude and sign of the angle will fix up the position of the other arm, and thereby will make the trigonometrical ratios unique.

10. Fundamental relations between the Trigonometrical ratios of any angle.

From the very definitions given in Art. 7 of the trigonometrical ratios of any angle XOP (=0) of whatever magnitude and sign, we at once derive the following relations:

cosec
$$\theta = \sin \theta$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$
and since $\sin \theta = \frac{PN}{OP}$, $\cos \theta = \frac{ON}{OP}$, $\tan \theta = \frac{PN}{ON}$, $\cot \theta = \frac{ON}{PN}$, we get $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$
.

Again, since in the right-angled triangle OPN,

$$OP^2 = PN^2 + ON^2.$$

dividing by OP^2 , ON^2 and PN^2 respectively, we get

$$\binom{OP}{ON}^2 = \binom{PN}{ON}^2 + 1 \qquad \cdots \qquad \cdots \qquad (ii)$$

$$\left(\frac{OP}{PN} \right)^2 = 1 + \left(\frac{ON}{PN} \right)^2 \cdot \qquad \cdots \qquad \cdots \qquad (iii)$$

From the definition of the trigonometrical ratios, (i) gives

$$(\sin \theta)^2 + (\cos \theta)^2 = 1.$$

Now, it is usual to write $(\sin \theta)^2$ in the form $\sin^2 \theta$ and so for other ratios. The relation then reduces to the form $\sin^2 \theta + \cos^2 \theta = 1$.

Similarly. (ii) and (iii) give respectively.

$$\sec^2\theta = 1 + \tan^2\theta$$
$$\csc^2\theta = 1 + \cot^2\theta.$$

These formulæ are also used in the forms

$$\sin^2\theta = 1 - \cos^2\theta$$
, $\cos^2\theta = 1 - \sin^2\theta$,
 $\sec^2\theta - \tan^2\theta = 1$, $\tan^2\theta = \sec^2\theta - 1$, etc.

Note. The fundamental formulæ derived in this article are very important, and are true for all values of θ whatever its magnitude and sign may be. For example, if we take $\frac{\theta}{2}$ in place of θ , we are simply taking a different angle for which the same relations are true, so that $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$, etc.

11. Conversions of Trigonometrical ratios.

With the help of the formulæ of the previous article, we can express any trigonometrical ratio of an angle in terms of any other ratio for the same angle; hence if the value of any trigonometrical ratio of an angle be given, we can find the value of any other ratio.

Ex. 1. Express $\sin \theta$ in terms of $\cot \theta$.

From the formulæ cosec
$$\theta = \frac{1}{\sin \theta}$$

and $\csc^2 \theta = 1 + \cot^2 \theta$,
we get $\sin \theta = \frac{1}{\cos \theta} = \frac{1}{\cos^2 \theta}$

Ex. 2. Express cosec θ in terms of sec θ .

cosec
$$\theta = \pm \sqrt{1 + \cot^2 \theta} = \pm \sqrt{1 + \frac{1}{\tan^2 \theta}}$$

$$= \pm \sqrt{\frac{\tan^2 \theta + 1}{\tan^2 \theta}} = \pm \sqrt{\frac{\sec^2 \theta}{\sec^2 \theta - 1}} = \frac{\pm \sec \theta}{\sqrt{\sec^2 \theta - 1}}$$

Ex. 3. If $\cos A = \frac{12}{18}$, find $\tan A$.

We have
$$\tan A = \frac{\sin A}{\cos A} = \pm \frac{\sqrt{1 - \cos^2 A}}{\cos A}$$

= $\pm \frac{\sqrt{1 - \frac{1 + \delta}{160}}}{\frac{1 + \delta}{18}} = \pm \frac{5}{12}$.

A more practical method in such cases is however to construct a right-angled triangle with the numerator and denominator as the two suitable sides, as shown below.

Ex. 4. If sec
$$A = \frac{41}{8}$$
, find cot A,

Let APN be a triangle right-angled at N in which the hypotenuse AP=41, AN=9,

so that
$$\sec NAP = \frac{AP}{AN} = \frac{41}{9}$$
.

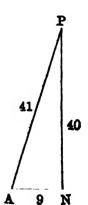
Thus $\angle NAP = A$.

Now,
$$PN^2 = AP^2 - AN^2 = 41^2 - 9^2$$

= 40^2 .

so that $PN = \pm 40$.

$$\therefore \quad \cot A = \cot NAP = \frac{AN}{PN} = \pm \frac{9}{40}.$$



12. Restrictions on the magnitudes of Trigonometrical ratios.

From the relation $\sin^2\theta + \cos^2\theta = 1$, since $\sin^2\theta$ and $\cos^2\theta$ being square quantities are both positive, it is evident that neither $\sin^2\theta$ nor $\cos^2\theta$ can exceed 1, for if $\sin^2\theta$, for example, be greater than 1, $\cos^2\theta$ (which is a square quantity) becomes negative, which is impossible. Thus, $\sin\theta$ as well as $\cos\theta$ must have numerical values not exceeding 1; in other words, both $\sin\theta$ and $\cos\theta$ must lie between +1 and -1 whatever the magnitude of θ may be. Any value numerically greater than 1, like -2 or +31 must be impossible for $\sin\theta$ or $\cos\theta$ so long θ is real.

sec θ and cosec θ therefore, being reciprocals of $\cos \theta$ and $\sin \theta$ respectively, can never be numerically less than 1.

tan θ and cot θ however, can have any numerical value greater than 1 or less than 1 according to the value of θ .

13. A few examples on the applications of the fundamental formulæ are given belew.

Ex. 1. Prove that
$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \csc\theta + \cot\theta$$
.

$$\begin{bmatrix} C. \ U. \ 1937 \end{bmatrix}$$

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \csc\theta + \cot\theta.$$

We have
$$\frac{1}{\sec A + \tan A} \frac{1}{\cos A} \frac{1}{\cos A} \frac{1}{\sec A - \tan A}$$

$$= \frac{1}{\sec A + \tan A} \frac{1}{\sec A - \tan A}$$

$$= \frac{\sec A - \tan A + \sec A + \tan A}{(\sec A + \tan A)(\sec A - \tan A)} \frac{2 \sec A}{\sec^2 A - \tan^3 A}$$

$$= 2 \sec A = \frac{2}{\cos A} \frac{1}{\cos A} \frac{1}{\cos A}$$

Hence, by transposition,

$$\frac{1}{\sec A + \tan A} \frac{1}{\cos A} \frac{1}{\cos A} \frac{1}{\sec A - \tan A}$$

Ex. 3. Prove that
$$\frac{1+2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta)(\cot \theta + \tan \theta)}$$
$$= \sin \theta \cos \theta (\sin \theta + \cos \theta).$$

We have
$$\frac{1 + 2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta)(\cot \theta + \tan \theta)} = \frac{(\sin^2 \theta + \cos^2 \theta) + 2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta) \left(\frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta}\right)} = \frac{(\sin \theta + \cos \theta)^2}{(\sin \theta + \cos \theta) \left(\frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta}\right)}$$

$$\frac{(\sin \theta + \cos \theta)}{(\sin \theta + \cos \theta)} = \frac{(\sin \theta + \cos \theta)}{(\sin \theta + \cos \theta)} = \frac{(\sin \theta + \cos \theta)}{(\sin \theta + \cos \theta)}$$

$$\frac{(\sin \theta + \cos \theta) \sin \theta \cos \theta}{1}$$

 $=\sin\theta\cos\theta(\sin\theta+\cos\theta).$

Ex. 4. If $15 \sin^2 \theta + 2 \cos \theta = 7$, find $\tan \theta$.

Here $15(1-\cos^2\theta)+2\cos\theta=7$,

whence $15 \cos^2 \theta - 2 \cos \theta - 8 = 0$.

or, $(5 \cos \theta - 4)(3 \cos \theta + 2) = 0$, $\therefore \cos \theta = \frac{4}{5}$, or, $-\frac{2}{3}$.

Case (i) when $\cos \theta = \frac{4}{6}$,

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{2}\frac{R}{5} = \frac{9}{25}$$
. $\therefore \sin \theta = \pm \frac{8}{5}$,

and so $\tan \theta = \frac{\sin \theta}{\cos \theta} = \pm \frac{3}{4}$.

Case (ii) when $\cos \theta = -\frac{2}{3}$,

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{4}{9} = \frac{\kappa}{9}. \qquad \therefore \quad \sin\theta = \pm \frac{\sqrt{5}}{9}.$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} - \frac{1}{2} \frac{\sqrt{5}}{2}$$

Examples II

Prove the following identities (Ex. 1 to 24):-

1.
$$\frac{\sin A + \cos A}{\sec A + \csc A} = \sin A \cos A.$$

2.
$$\cot \theta + \tan \theta = \sec \theta \csc \theta$$
.

3.
$$\frac{1}{1+\tan A} = \frac{\cot A}{1+\cot A}$$
.

4.
$$\csc^6 A - \cot^6 A = 1 + 3 \csc^2 A \cot^2 A$$
.

5.
$$\cos^6 A + \sin^6 A = 1 - 3 \sin^2 A \cos^2 A$$
.

6.
$$\frac{1}{\cos^2 A} - \frac{1}{\csc^2 A - 1} = 1$$
.

7.
$$\cos A + \tan A \sin A = \sec A$$
.

8.
$$\sec^4 A + \tan^4 A = 1 + 2 \sec^2 A \tan^2 A$$
.

9.
$$\frac{1+3\cos\theta-4\cos^3\theta}{1-\cos\theta}=(1+2\cos\theta)^2.$$

10.
$$(\cot \theta + \csc \theta)^2 = \frac{1 + \cos \theta}{1 - \cos \theta}$$

11.
$$\frac{1+\tan^2\theta}{1+\cot^2\theta} = \left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2.$$

12.
$$\frac{\tan^{2} a - \cot^{2} a}{1 + \cot^{2} a} = \frac{\sin^{2} a - \cos^{2} a}{\cos^{2} a}.$$

13.
$$1 + \tan \theta + \sec \theta = \frac{2}{1 + \cot \theta - \csc \theta}$$

14.
$$\frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \frac{1 + \sin \theta}{\cos \theta}$$

15.
$$\frac{\sin A - 2 \sin^3 A}{2 \cos^3 A - \cos A} = \tan A$$
.

16.
$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$
 sec $\theta = \sec\theta - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

17.
$$\frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A - \cot A} = \frac{\sin^2 A}{(1 - \cos A)^2}.$$

18.
$$(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A)$$

19.
$$\frac{\sec \theta + \tan \theta}{\csc \theta + \cot \theta} - \frac{\sec \theta - \tan \theta}{\csc \theta - \cot \theta} = 2(\sec \theta - \csc \theta).$$

20.
$$\frac{1}{1+\sin^2\theta} + \frac{1}{1+\csc^2\theta} = 1$$
.

21.
$$\frac{\sin^3 a + \cos^3 a}{\sin a + \cos a} + \frac{\sin^3 a - \cos^3 a}{\sin a - \cos a} = 2.$$

22.
$$\frac{\tan \theta}{\sec \theta - 1} - \frac{\sin \theta}{1 + \cos \theta} = 2 \cot \theta.$$

23.
$$\cos \theta + \cos \phi = \sin \theta + \sin \phi$$

 $\sin \theta - \sin \phi = \cos \phi - \cos \theta$

24.
$$1+4 \operatorname{cosec}^2 \theta \cot^2 \theta = (\operatorname{cosec}^2 \theta + \cot^2 \theta)^2$$
.

- 25. Express $1-2 \sin \theta \cos \theta$ as a perfect square.
- 26. Express $2 \sec^2 \theta \sec^4 \theta 2 \csc^2 \theta + \csc^4 \theta$ in terms of $\tan \theta$.
 - 27. Prove that

(sin
$$a \cos \beta + \cos a \sin \beta$$
)(sin $a \cos \beta - \cos a \sin \beta$)
$$= \sin^2 a - \sin^2 \beta$$

- 28. If $\sin A + \sin^2 A = 1$, then $\cos^2 A + \cos^4 A = 1$.
- 29. (i) If $\sin \theta \cos \theta = 0$, prove that $\sec \theta = \pm \sqrt{2}$.
 - (ii) If $7 \sin^2 \theta + 3 \cos^3 \theta = 4$, show that $\tan \theta \pm \frac{1}{\sqrt{3}}$.
 - (iii) If 3 sin $\theta + 4 \cos \theta = 5$, show that sin $\theta = \frac{3}{8}$.

30. If
$$\tan \theta + \sec \theta = x$$
, show that $\sin \theta = \frac{x^3 - 1}{x^2 + 1}$.

31. If
$$\tan \theta = \frac{a}{b}$$
, find the value of $\frac{a \sin \theta - b \cos \theta}{a \sin \theta + b \cos \theta}$

- 32. If $1 + 4x^2 = 4x \sec A$, prove that $\sec A + \tan A = 2x \text{ or } 1/2x$.
- 33. Express $\sin \alpha$ in terms of $\sec \alpha$, and $\sec \theta$ in terms of $\cot \theta$.
- 34. Given $\sin \theta = \frac{8}{5}$, $\cos \phi = \frac{12}{15}$, where θ and ϕ are acute angles, find the value of $\tan \theta \tan \phi$.
 - 35. If $\cos a + \sin a = \sqrt{2} \cos a$, prove that $\cos a \sin a = \sqrt{2} \sin a$.
 - 36. If $\tan A = \frac{1}{\sqrt{3}}$, find $\frac{\csc^2 A \sec^2 A}{\csc^2 A + \sec^2 A}$.
 - 37. If $1 + \sin^2 A = 3 \sin A \cos A$, find tan A.
 - 38. If $\tan \theta + \sin \theta = m$, $\tan \theta \sin \theta = n$, prove that $m^2 n^2 = 4 \sqrt{mn}$.
- 39. If $(a^2 b^2) \sin \theta + 2ab \cos \theta = a^2 + b^2$, find $\tan \theta$ and $\csc \theta$.
 - 40. If $\tan \theta = \frac{\sin a \cos a}{\sin a + \cos a}$, prove that $\sqrt{2} \cos \theta = \sin a + \cos a$.
 - 41. If $\cos^4\theta + \cos^2\theta = 1$, show that
 - (i) $\tan^4 \theta + \tan^2 \theta = 1$.
 - (ii) $\cot^4 \theta \cot^2 \theta = 1$.
- 42. If x and y are two unequal real quantities, show that the equations (i) $\sin^2\theta = \frac{(x+y)^2}{4xy}$ and (ii) $\cos\theta = x + \frac{1}{x}$ are both impossible.
 - 43. Eliminate θ between
 - (i) $x = a \cos \theta$, $y = b \sin \theta$.
 - (ii) $x = c (\sec \theta + \tan \theta), y = c (\sec \theta \tan \theta).$
 - (iii) $a \cos \theta + b \sin \theta + c = 0$, $a' \cos \theta + b' \sin \theta + c' = 0$.
 - (iv) $a \tan^2 \theta + b \tan \theta + c = a' \cot^2 \theta + b' \cot \theta + c' = 0$.

Examples II(A)

Prove the following identities (Ex. 1 to 18):—

1.
$$\frac{\tan^3 a}{1 + \tan^3 a} + \frac{\cot^3 a}{1 + \cot^2 a} = \frac{1 - 2\sin^2 a \cos^2 a}{\sin a \cos a}.$$

2.
$$(\tan \theta + \cot \theta + \sec \theta)(\tan \theta + \cot \theta - \sec \theta) = \csc^2 \theta$$
.

3.
$$\sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta) = \sec \theta + \csc \theta$$
.

[C. U 1935]

4.
$$(1 + \sin \alpha - \cos \alpha)^2 + (1 - \sin \alpha + \cos \alpha)^2$$

- $4(1 - \sin \alpha \cos \alpha)$.

5.
$$\sin^6 a + \sin^4 a \cos^2 a - \sin^2 a \cos^4 a - \cos^6 a$$

= $\sin^2 a - \cos^2 a$.

6.
$$3(\sin\theta + \cos\theta) - 2(\sin^3\theta + \cos^3\theta) = (\sin\theta + \cos\theta)^3$$
.

7.
$$\frac{1}{\operatorname{cosec} \theta - \cot \theta} - \frac{1}{\sin \theta} - \frac{1}{\sin \theta} - \frac{1}{\operatorname{cosec} \theta + \cot \theta}$$

8.
$$\frac{\cos x}{\sin x + \cos y} + \frac{\cos y}{\sin y - \cos x} = \frac{\cos x}{\sin x - \cos y} + \frac{\cos y}{\sin y + \cos x}$$

9.
$$(\sin \theta + \csc \theta)^2 + (\cos \theta + \sec \theta)^2 = \tan^2 \theta + \cot^2 \theta + 7$$
.

10.
$$(\sec \theta - \cos \theta)(\csc \theta - \sin \theta)(\tan \theta + \cot \theta) = 1$$
.

11.
$$\frac{1 + (\cos c \ x \tan y)^2}{1 + (\cos c \ z \tan y)^2} = \frac{1 + (\cot x \sin y)^2}{1 + (\cot z \sin y)^2}$$

12.
$$\sec^3 a \csc^3 a - 3 \sec a \csc a = \tan^3 a + \cot^3 a$$
.

13.
$$\sin^6 A - \cos^6 A = (\sin A + \cos A)(\sin A - \cos A)$$

 $\times (1 + \sin A \cos A)(1 - \sin A \cos A).$

14.
$$\frac{\tan \alpha}{(1+\tan^2 \alpha)^2} + \frac{\cot \alpha}{(1+\cot^2 \alpha)^2} = \sin \alpha \cos \alpha$$
.

15.
$$\sin^2\theta \tan \theta - \cos^2\theta \cot \theta + \sec \theta \csc \theta = 2 \tan \theta$$
.

16.
$$\frac{\cos^2 A - \sin^2 A}{\sin A \cos^2 A - \cos A \sin^2 A} = \csc A + \sec A.$$

17.
$$\frac{\tan^2 A + \cot^2 A}{\tan^2 A - \cot^2 A} = \frac{\sin^4 A + \cos^4 A}{\sin^2 A - \cos^2 A}$$

- 18. $(\sin \alpha \cos \beta \cos \alpha \sin \beta)^2 + (\cos \alpha \cos \beta + \sin \alpha \sin \beta)^2 = 1.$
- 19. If $\cos^2 A \sin^2 A = \tan^2 B$, then $\cos^2 B - \sin^2 B = \tan^2 A$.
- 20. If $\sin^4 x + \sin^2 x = 1$, the $\tan^4 x \tan^2 x = 1$.
- 21. Show that the difference between $3 \sin^4 \theta 2 \sin^6 \theta$ and $2 \cos^6 \theta 3 \cos^4 \theta$ is the same for all values of θ .
 - 22. If $x = \frac{1 + \sin \theta}{\cos \theta}$, show that $\frac{1}{x} = \frac{1 \sin \theta}{\cos \theta}$.
 - 23. If $\tan^2 A = 1 + 2 \tan^2 B$, show that $\cos^2 B = 2 \cos^2 A$.
 - 24. If $\sin a + \cos a = 1$, the $\sin a \cos a = \pm 1$.
 - 25. If $a \cos \theta b \sin \theta = c$, then show that $a \sin \theta + b \cos \theta = \pm \sqrt{a^2 + b^2 c^2}$.
 - 26. If $(1 + \sin x)(1 + \sin y)(1 + \sin z)$ $= (1 - \sin x)(1 - \sin y)(1 - \sin z)$,

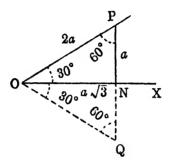
prove that each is equal to $\pm \cos x \cos y \cos z$.

- 27. If $x \sin^8 a + y \cos^8 a = \sin a \cos a$, and $x \sin a y \cos a = 0$, then $x^2 + y^2 = 1$. [C. U. 1937]
- 28. If $\sin A = \frac{\sin x + \sin y}{1 + \sin x \sin y}$, show that $\cos A = \pm \frac{\cos x \cos y}{1 + \sin x \sin y}.$
- 29. (i) If $\sin a + \csc a = 2$, then $\sin^n a + \csc^n a = 2$.
 - (ii) If $\sec \alpha = \sec \beta \sec \gamma + \tan \beta \tan \gamma$, then $\sec \beta = \sec \gamma \sec \alpha \pm \tan \gamma \tan \alpha$.
- 30. If $\frac{\cos^4 x}{\cos^2 y} + \frac{\sin^4 x}{\sin^2 y} = 1$, then $\frac{\cos^4 y}{\cos^2 x} + \frac{\sin^4 y}{\sin^2 x} = 1$.

CHAPTER III

TRIGONOMETRICAL RATIOS OF SOME STANDARD ANGLES

14. Ratios of 30°.



Let the angle XOP, which may be supposed to be traced out by a revolving line starting from OX, be 30°. Let PN be drawn perpendicular upon OX from any point P on OP. The angle OPN is then 60°.

Produce PN to Q, making NQ = NP. Join OQ. The triangles PON and QON are easily seen to be equal in all respects, and so $\angle OQN = \angle OPN = 60^{\circ}$. Hence, the triangle OPQ is equilateral, and so OP = PQ = double of PN.

Hence, in the above figure, if PN=a, then OP=2a and so $ON=\sqrt{OP^2-PN^2}=\sqrt{4a^2-a^2}=\sqrt{3a}$. The sides ON, PN and OP are all positive in this case, since the angle is acute.

Hence,

$$\sin 30^{\circ} - \sin PON = \frac{PN}{OP} - \frac{a}{2a} - \frac{1}{2}$$

$$\cos 30^{\circ} = \frac{ON}{OP} - \frac{\sqrt{3}a}{2a} - \frac{\sqrt{3}}{2}$$

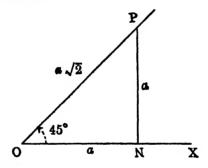
$$\tan 30^{\circ} = \frac{PN}{ON} = \frac{1}{\sqrt{3}}$$

$$\cot 30^{\circ} = \frac{ON}{PN} = \sqrt{3}$$

$$\csc 30^{\circ} = \frac{1}{\sin 30^{\circ}} = 2$$

$$\sec 30^{\circ} = \frac{1}{\cos 30^{\circ}} = \frac{2}{\sqrt{3}}$$

15. Ratios of 45°.



Let $\angle XOP = 45^{\circ}$. PN is perpendicular on OX. In the right-angled triangle PON, $\angle PON = 45^{\circ}$.

Therefore, $\angle OPN$ is also 45° and so ON = PN = a suppose. Then $OP = \sqrt{ON^2 + PN^2} = \sqrt{a^2 + a^2} = a\sqrt{2}$.

Hence,

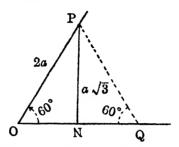
$$\sin 45^{\circ} = \frac{PN}{OP} - \frac{1}{\sqrt{2}}$$

$$\cos 45^{\circ} = \frac{ON}{OP} - \frac{1}{\sqrt{2}}$$

$$\tan 45^{\circ} = \frac{PN}{ON} - 1$$

$$\sec 45^{\circ} = \csc 45^{\circ} = \sqrt{2}, \cot 45^{\circ} = 1.$$

16. Ratios of 60°.



Let $\angle XOP = 60^{\circ}$. Now PN being perpendicular upon OX, along NX out off NQ = ON. Join PQ. Then the two triangles OPN and QPN are easily seen to be congruent. Hence, $\angle PQN = \angle PON = 60^{\circ}$. Thus, the triangle POQ is equilateral, and so OP = OQ = double of ON.

If ON = a, then OP = 2a and hence $PN = \sqrt{OP^2 - ON^2} = a\sqrt{3}$.

Then
$$\sin 60^{\circ} \cdot \frac{PN}{OP} = \frac{\sqrt{3}}{2}$$

$$\cos 60^{\circ} \cdot \frac{ON}{OP} = \frac{1}{2}$$

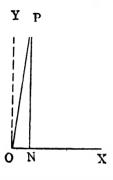
$$\tan 60^{\circ} \cdot \frac{PN}{ON} = \sqrt{3}$$

$$\cot 60^{\circ} = \frac{1}{\sqrt{3}}$$
, $\sec 60^{\circ} = 2$, $\csc 60^{\circ} = \frac{2}{\sqrt{3}}$.

Note. It may be noted from the values of the ratios that $\sin 60^\circ = \cos 30^\circ$, $\cos 60^\circ = \sin 30^\circ$, $\tan 60^\circ = \cot 30^\circ$, $\cot 60^\circ = \tan 30^\circ$, $\sec 60^\circ = \csc 30^\circ$, $\csc 60^\circ = \sec 30^\circ$. It will be proved more generally, in the next chapter, that for any two complementary angles sine of one is the cosine of the other and *vice versa*, tangent of one is the cotangent of the other, and secant of one is the cosecant of the other. The angle 45° being its own complement, therefore, it should have its sine and cosine equal to one another, as is actually seen to be the case.

17. Ratios of 90°.

Let XOP be an acute angle very nearly 90°. PN being



perpendicular upon OX, ON is extremely small, and as $\angle XOP$ approaches more and more to 90° , ON becomes smaller and smaller. The length OP may however remain finite, and PN and OP will approach each other more and more closely. Ultimately when $\angle XOP$ becomes 90° , OP and PN coincide, and ON becomes zero ultimately. Hence the ratio

PN/OP becomes 1 and ON/OP becomes zero.

Thus,
$$\sin 90^{\circ} = \frac{PN}{OP}$$
 in the limit = 1
$$\cos 90^{\circ} = \frac{ON}{OP}$$
 in the limit = 0
$$\tan 90^{\circ} = \frac{PN}{ON}$$
 in the limit = ∞ * (infinity)
(since $ON \to 0$, whereas PN remains finite)
$$\cot 90^{\circ} = \frac{\cos 90^{\circ}}{\sin 90^{\circ}} = \frac{0}{1} = 0$$

$$\csc 90^{\circ} = 1, \sec 90^{\circ} = OP/ON \text{ in the limit} = 0$$

*The symbol ∞ is used to denote a quantity which exceeds any positive number, however large, and does not represent a definite number.

It should be noted that in determining $\tan 90^\circ$, we may start with an angle XOP, slightly greater than 90° , (i.e., in the second quadrant), and make it approach 90° . Then ON will be negative and $\to 0$, whereas PN is positive, Accordingly we may also write $\tan 90^\circ = -\infty$. Thus strictly speaking, we should write $\tan 90^\circ = \pm \infty$. Similar remarks apply for see 90° , oot 0° , cose 0° .

18. Ratios of 0°.



Let $\angle XOP$ be an infinitely small positive angle, and let PN be perpendicular on OX.

Then, PN is infinitely small, whereas OP is finite. Now, if $\angle XOP$ be taken less and less and ultimately becomes less than any quantity we can assign, we denote it by zero, and in this case PN practically vanishes, whereas OP and ON remaining finite, coincide. Hence, the ratio PN/OP becomes ultimately zero, and ON/OP becomes 1.

Hence,
$$\sin 0^{\circ} = \frac{PN}{OP}$$
 in the limit = 0
$$\cos 0^{\circ} = \frac{ON}{OP} \text{ in the limit} = 1$$

$$\tan 0^{\circ} = \frac{\sin 0^{\circ}}{\cos 0^{\circ}} = \frac{0}{1} = 0$$

$$\cot 0^{\circ} = \frac{ON}{PN} \text{ in the limit} = \infty^{*}$$

$$\csc 0^{\circ} = \frac{OP}{PN} \text{ in the limit} = \infty^{*}$$

$$\sec 0^{\circ} = \frac{1}{\cos 0^{\circ}} = \frac{1}{1} = 1.$$

Note. Note that 0° and 90° being complementary, sin 0°=cos 90°, cos 0°=sin 90°=1, etc.

19. As the ratios of the standard angles 0°, 30°, 45°, 60°, and 90° are very often used, they should be remembered very

^{*} See foot note of Art. 17.

carefully. The first three ratios are given in the tabulated form below. The other three are reciprocals to these.

angle	sine	cosine	tangent
0° or 0°	0	1	0
30° or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45° or $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60° or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	√3
90° or $\frac{\pi}{2}$	1	0	± ∞

Note. The following device may be of use in remembering the sines and cosines of standard angles. The sines of the angles 0°, 30°, 45°, 60°, 90° are respectively the square roots of the fractions

and cosines of these angles are the square roots from right to left.

20. Examples worked out.

Ex. 1. If
$$\theta = 30^{\circ}$$
, verify that $\cos 2\theta = \frac{1 - \tan^3 \theta}{1 + \tan^3 \theta}$

Here,
$$\cos 2\theta = \cos 60^\circ = \frac{1}{2}$$
. Also $\tan \theta = \tan 30^\circ = \frac{1}{\sqrt{3}}$.

$$\therefore \frac{1-\tan^{2}\theta}{1+\tan^{2}\theta} = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{1}{2}.$$

Hence,
$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

Ex. 2. Verify that

$$\sin 30^{\circ} - \sin 60^{\circ} \cos 30^{\circ} - \cos 60^{\circ} \sin 30^{\circ}$$
.

The right-hand side, on substitution of the values,

$$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}=\sin 30^{\circ}.$$

Hence the result.

Ex. 3. Solve for θ , where θ is a positive acute angle, given cosec θ cot $\theta = 2\sqrt{3}$.

From the given equation, $\frac{1}{\sin \theta} \frac{\cos \theta}{\sin \theta} = 2\sqrt{3}$,

or,
$$\cos \theta = 2 \sqrt{3} \sin^2 \theta = 2 \sqrt{3} (1 - \cos^2 \theta)$$
,

whence, $2\sqrt{3}\cos^2\theta + \cos\theta - 2\sqrt{3} = 0$

giving
$$\cos \theta = \frac{-1 \pm \sqrt{1+48}}{4\sqrt{3}} = \frac{-1 \pm 7}{4\sqrt{3}}$$
.

Since θ is a positive acute angle, $\cos \theta$ is positive, and so rejecting the negative value,

$$\cos \theta = \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2} = \cos 30^{\circ}. \quad \therefore \quad \theta = 30^{\circ} \quad i.e., \quad \frac{\pi}{6}$$

Examples III

Verify the results (Ex. 1 to 6):

1.
$$1-2\sin^2 30^\circ - 2\cos^2 30^\circ - 1 = \cos 60^\circ$$
.

2.
$$\frac{2 \tan 30^{\circ}}{1 - \tan^2 30^{\circ}} = \sqrt{3}$$
.

3.
$$\cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} - \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6}$$

4. (i)
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii)
$$\cos A = \cos^2 B - \sin^2 B$$
, where $A = 60^{\circ}$, $B = 30^{\circ}$.

5.
$$\sin 3A = 3 \sin A - 4 \sin^3 A$$
, where $A = \frac{\pi}{6}$.

6.
$$\cos^2 45^\circ$$
. $\sec^2 30^\circ (\sin^8 30^\circ + 4 \cot^2 45^\circ - \sec^2 60^\circ) = \frac{1}{8}$.

7. If
$$\tan^2 \frac{\pi}{4} - \cos^2 \frac{\pi}{3} = x \sin \frac{\pi}{4} \cos \frac{\pi}{4} \tan \frac{\pi}{3}$$
, find x.

- 8. If θ be a positive acute angle, find θ , when
 - (i) $2 \sin^2 \theta = 3 \cos \theta$.
 - (ii) $\tan \theta + \cot \theta = 2$.
 - (iii) $\csc^2\theta + 5 = 3\sqrt{3} \cot \theta$.
 - (iv) $\sin \theta + \cos \theta = \sqrt{2}$.
- ' (v) $2(\cos^2\theta \sin^2\theta) = 1$.
 - (vi) $6 \sin^2 \theta 11 \sin \theta + 4 = 0$.

(vii)
$$\frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}}$$

- 9. Given θ and ϕ to be positive acute angles, and $\tan (\theta + \phi) = \sqrt{3}$, $\tan (\theta \phi) = 1$, determine θ and ϕ .
 - 10. Find α and β (α and β being positive acute angles), if $\sin (2\alpha \beta) = 1$,

and $\cos(\alpha+\beta)=\frac{1}{2}$.

11. Find A, B, C(A, B, C being positive acute angles), if $\sin (B+C-A)=1,$ $\cos (C+A-B)=1,$ and $\tan (A+B-C)=1.$

12. Find the numerical values of :-

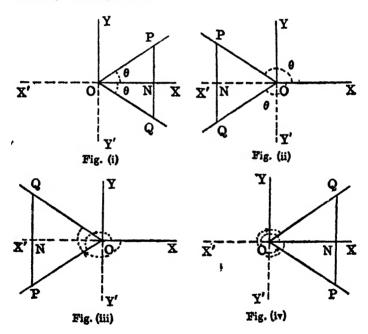
(i)
$$\cot^2 \frac{\pi}{6} - 2 \cos^2 \frac{\pi}{3} - \frac{3}{4} \sec^2 \frac{\pi}{4} - 4 \sec^2 \frac{\pi}{6}$$

(ii) $3 \tan^2 45^\circ - \sin^2 60^\circ - \frac{1}{2} \cot^2 30^\circ + \frac{1}{8} \sec^2 45^\circ$.

CHAPTER IV

TRIGONOMETRICAL RATIOS OF ANGLES ASSOCIATED WITH A GIVEN ANGLE θ

21. Ratios of the angle $(-\theta)$ in terms of those of θ , θ having any magnitude.



Let the $\angle XOP$ be θ and the $\angle XOQ$ described clockwisely be $-\theta$. From any point P on OP draw PN perpendicular to OX [or OX' as in Figs. (ii) and (iii)], and produce it to meet OQ at Q say.

Now, $\angle XOP$ (measured anti-clockwisely) being equal to $\angle XOQ$ (measured clockwisely), $\angle PON = \angle QON$ in magnitude in all the figures, and therefore, the two rt.-angled triangles PON and QON are congruent. The corresponding sides are therefore equal in magnitude. Considering the signs of these sides according to the usual convention, we get in all the figures,

$$QN = -PN$$
, and $OQ = OP$.

(both OP and QQ being always considered positive)

Hence, from definition,

$$\sin (-\theta) = \frac{QN}{OQ} = \frac{-PN}{OP} = -\sin \theta$$

$$\cos (-\theta) = \frac{ON}{OQ} = \frac{ON}{OP} = \cos \theta$$

$$\tan (-\theta) = \frac{QN}{ON} = \frac{-PN}{ON} = -\tan \theta$$

and the reciprocals of these give,

$$cosec(-\theta) = -cosec\theta$$
,
 $sec(-\theta) = sec\theta$,
 $cot(-\theta) = -cot\theta$.

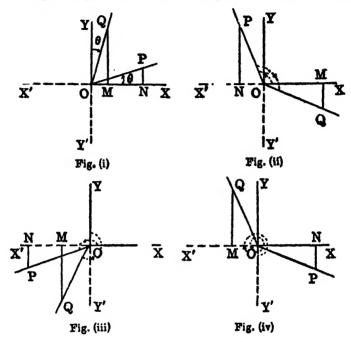
22. Ratios of $(90^{\circ} - \theta)$.

Let the $\angle XOP$ traced out by a revolving line be θ , and let another revolving line, starting from OX trace out the angle $XOY = 90^{\circ}$ and then revolve back, tracing out $\angle YOQ = \theta$ in the clockwise direction, so that $\angle XOQ = 90^{\circ} - \theta$.

Take two equal lengths OP and OQ along OP and OQ respectively and draw PN and QM perpendiculars on OX.

If OP be in the first or third quadrant as in Fig. (i) and Fig. (ii), OQ also lies in the same quadrant. If OP lies in the second quadrant as in Fig. (ii), OQ lies in the fourth quadrant; and if OP lies in the fourth, OQ lies in the

second, as in Fig. (iv). Now, $\angle XOP$ being equal to $\angle YOQ$ in magnitude, $\angle PON = \angle OQM$, and since OP = OQ,



the two right-angled triangles PON, OQM are congruent. The corresponding sides are therefore equal in magnitude. Considering signs as well, we get in all the figures,

$$QM = ON$$
, $OM = PN$, $OQ = OP$.

Hence, from definition,

$$\sin (90^{\circ} - \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{ON}{OP} = \cos \theta$$

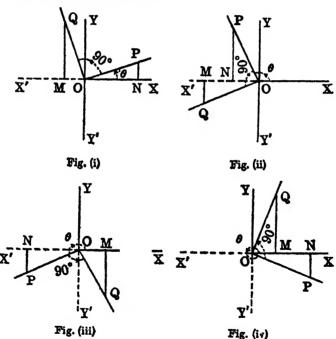
$$\cos (90^{\circ} - \theta) = \frac{OM}{OQ} = \frac{PN}{OP} = \sin \theta$$

$$\tan (90^{\circ} - \theta) = \frac{QM}{OM} = \frac{ON}{PN} = \cot \theta.$$

The reciprocals of these are
$$\csc (90^{\circ} - \theta) = \sec \theta$$
, $\sec (90^{\circ} - \theta) = \csc \theta$, $\cot (90^{\circ} - \theta) = \tan \theta$.

Obs. The angles $(90^{\circ}-\theta)$ is the complement of θ , and we derive the result that for a pair of complementary angles, since of one is the cosine of the other and vice versa, tangent of one is the cotangent of the other and secant of one is the cosecant of the other. This was verified in the last chapter in connection with the complementary pairs 30° and 60°, as also 0° and 90°.

23. Ratios of $(90^{\circ}+6)$.



Let a revolving line, starting from OX, trace out an $\angle XOP = \theta$, and further trace out an $\angle POQ = 90^{\circ}$, so that $\angle XOQ = 90^{\circ} + \theta$.

Cut off OP = OQ along OP and OQ respectively and let PN, QM be perpendiculars on OX (produced where necessary).

Now, OQ being perpendicular to OP, the $\angle PON$ = the complement of $\angle QOM = \angle OQM$ in magnitude, and since OP = OQ, the two right-angled triangles OPN and OQM are congruent. The corresponding sides are therefore equal. Considering signs as well, we get, for all the figures.

$$QM = ON$$
, $OM = -PN$, $OQ = OP$.

Hence, from definition,

$$\sin (90^{\circ} + \theta) - \sin \angle XOQ = \frac{QM}{OQ} - \frac{ON}{OP} - \cos \theta$$

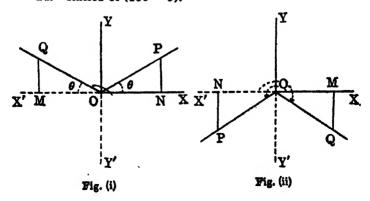
$$\cos (90^{\circ} + \theta) = \frac{OM}{OQ} = \frac{-PN}{OP} = -\sin \theta$$

$$\tan (90^{\circ} + \theta) = \frac{QM}{OM} = \frac{ON}{-PN} = -\cot \theta$$

and considering their reciprocals,

cosec
$$(90^{\circ} + \theta) = \sec \theta$$
,
sec $(90^{\circ} + \theta) = -\csc \theta$,
cot $(90^{\circ} + \theta) = -\tan \theta$.

24. Ratios of $(180^{\circ} - \theta)$.



Let $\angle XOP = \theta$ be traced out by a revolving line, and let another revolving line, starting from OX, trace out an angle 180° coming up to OX' and then revolve back and describe an angle $X'OQ = \theta$, so that $\angle XOQ = 180^{\circ} - \theta$.

Two figures are given here, one with OP in the first quadrant and another with OP in the third quadrant. The two other figures may easily be drawn by the students.

Now cut off OP = OQ, and draw PN and QM perpendiculars on OX (or OX' as the case may be). Then $\angle PON = \angle QOM$ in magnitude, and OP = OQ. Hence, the right-angled triangles PON and QOM are congruent, and so have their corresponding sides equal in magnitude. Taking into consideration the signs, we get for all the figures,

$$QM = PN$$
, $OM = -ON$, $OQ = OP$.

Hence, for all values of θ ,

$$\sin (180^{\circ} - \theta) = \sin XOQ = \frac{QM}{OQ} = \frac{PN}{OP} = \sin \theta$$

$$\cos (180^{\circ} - \theta) = \frac{OM}{OP} = \frac{-ON}{OP} = -\cos \theta$$

$$\tan (180^{\circ} - \theta) = \frac{QM}{OM} = \frac{PN}{ON} = -\tan \theta$$

and so taking reciprocals,

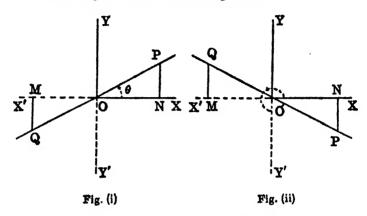
cosec
$$(180^{\circ} - \theta) = \text{cosec } \theta$$
,
sec $(180^{\circ} - \theta) = - \text{sec } \theta$,
cot $(180^{\circ} - \theta) = - \text{cot } \theta$.

Note. The first two formulæ may be expressed in the form "sines of supplementary angles are equal, and cosines of supplementary angles are equal in magnitude but opposite in sign".

25. Ratios of $(180^{\circ} + \theta)$.

Let a revolving line starting from OX, trace out an angle $XOP = \theta$, and further trace out an angle $POQ = 180^{\circ}$, so that $\angle XOQ = 180^{\circ} + \theta$.

OP and OQ are then in one straight line.



Cut off OP = OQ, and draw PN and QM perpendiculars on XOX'.

Two figures are given here with OP in the first and fourth quadrants, and the other two may be similarly drawn.

Now, POQ being a straight line in this case, $\angle PON = \angle QOM$ in magnitude. Also, OP = OQ. Hence, the right-angled triangles PON and QOM are congruent and have their corresponding sides equal in magnitude. Considering signs, we get in all cases,

$$QM = -PN$$
, $OM = -ON$, $OQ = OP$.

Thus, for all values of θ ,

$$\sin (180^{\circ} + \theta) = \sin XOQ :: \frac{QM}{OQ} = \frac{-PN}{OP} = -\sin \theta$$

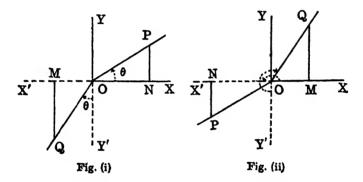
$$\cos (180^{\circ} + \theta) = \frac{OM}{OQ} = \frac{-ON}{OP} = -\cos \theta$$

$$\tan (180^{\circ} + \theta) = \frac{QM}{OM} = \frac{-PN}{-QN} = \frac{PN}{ON} = \tan \theta$$

and so.

cosec
$$(180^{\circ} + \theta) = -$$
 cosec θ ,
sec $(180^{\circ} + \theta) = -$ sec θ ,
cot $(180^{\circ} + \theta) =$ cot θ .

26. Ratios of $(270^{\circ} - \theta)$.



Let $\angle XOP = \theta$ be traced out by a revolving line, and let another revolving line trace out an angle $XOY' = 270^{\circ}$, thereby coming up to the position OY', and then revolve back, tracing out an angle $Y'OQ = \theta$, so that $\angle XOQ = 270^{\circ} - \theta$.

Two figures are given here with OP in the first and third quadrants. The other two may be drawn similarly.

Cut off OP = OQ and draw PN, QM perpendiculars on XOX'.

Since $\angle XOP = \angle Y'OQ$ in magnitude, we easily derive that $\angle PON = \angle OQM$ in magnitude. Also OP = OQ. Hence, the two right-angled triangles OPN and OQM are congruent. Considering signs, we get for all the figures,

$$QM = -ON$$
, $OM = -PN$, $OQ = OP$.

Hence, for all values of θ ,

$$\sin (270^{\circ} - \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{-ON}{OP} = -\cos \theta$$

$$\cos (270^{\circ} - \theta) = \frac{OM}{OQ} = \frac{-PN}{OP} = -\sin \theta;$$

$$\tan (270^{\circ} - \theta) = \frac{QM}{OM} = \frac{-ON}{-PN} = \frac{ON}{PN} = \cot \theta;$$

and thus.

$$cosec (270^{\circ} - \theta) = -sec \theta,$$

 $sec (270^{\circ} - \theta) = -cosec \theta,$
 $cot (270^{\circ} - \theta) = tan \theta.$

27. Ratios of $(270^{\circ} + \theta)$.

We may proceed geometrically as in the previous cases. Otherwise we may proceed as follows:

$$\sin (270^{\circ} + \theta) = \sin (180^{\circ} + 90^{\circ} + \theta) = -\sin (90^{\circ} + \theta) [from \S 25$$

$$= -\cos \theta \qquad \cdots \qquad [from \S 23]$$

$$\cos (270^{\circ} + \theta) = \cos (180^{\circ} + 90^{\circ} + \theta) = -\cos (90^{\circ} + \theta)$$

$$= -(-\sin \theta) = \sin \theta$$

$$\tan (270^{\circ} + \theta) = \frac{\sin (270^{\circ} + \theta)}{\cos (270^{\circ} + \theta)} = \frac{-\cos \theta}{\sin \theta} = -\cot \theta;$$

and hence.

cosec
$$(270^{\circ} + \theta) = -\sec \theta$$
,
 $\sec (270^{\circ} + \theta) = \csc \theta$,
 $\cot (270^{\circ} + \theta) = -\tan \theta$.

Note. The ratios of $180^{\circ} - \theta$, $180^{\circ} + \theta$, $270^{\circ} - \theta$ can be similarly deduced from the formulæ for ratios of $90^{\circ} \pm \theta$.

28. Ratios of
$$(360^{\circ} - \theta)$$
, $(360^{\circ} + \theta)$ and $(n. 360^{\circ} \pm \theta)$.

It has already been remarked in Art. 2. Chapter I, that angles which differ by complete multiples of 360°, i.e., by an exact number of complete revolutions, have the final positions of the revolving lines coincident, if the initial lines

are the same. Hence, all the trigonometrical ratios of two such angles must be identical in magnitude as well as in sign.

Thus, trigonometrical ratios of $360^{\circ} - \theta$ must be same as those of $-\theta$. Hence,

$$\sin (360^{\circ} - \theta) = \sin (-\theta) = -\sin \theta$$

$$\cos (360^{\circ} - \theta) = \cos (-\theta) = \cos \theta$$

$$\tan (360^{\circ} - \theta) = \tan (-\theta) = -\tan \theta, \text{ etc.}$$

Trigonometrical ratios of $360^{\circ} + \theta$, or of $360^{\circ} \times n \pm \theta$, where *n* is an integer, positive or negative, must similarly be same as those of θ , or of $\pm \theta$.

Thus, in determining trigonometrical ratios of angles, complete multiples of 360° (i.e., 2π) may be always added or subtracted.

29. All the above results may, for easy remembrance, be summed up in a simple rule.

If θ be associated with an even multiple of 90° by + or - sign, $(e.g., 180^{\circ} - \theta, 180^{\circ} + \theta, 360^{\circ} - \theta, 360^{\circ} + \theta, \text{ etc.})$ the ratio is not altered in form (i.e., sine remains sine, cosine remains cosine, etc.). To determine the sign, assuming θ to be acute, find out the quadrant in which the associated angle lies, and determine the sign according to the rule, "all, sin, tan, cos".

If θ be associated with an odd multiple of 90° by + or - sign, $(e.g., 90^{\circ} - \theta, 90^{\circ} + \theta, 270^{\circ} - \theta, 270^{\circ} + \theta,$ etc.) the ratio is altered (sine becomes cosine, cosine becomes sine, tangent becomes cotangent, etc.). Moreover, the sign of the result is determined as in the previous paragraph.

Example. Consider formulæ for $\tan (270^{\circ} - \theta)$ and $\sec (180^{\circ} + \theta)$.

$$270^{\circ} - \theta = 3.90^{\circ} - \theta$$
 (multiple of 90° is odd).

Hence, the ratio will be altered, tan changing into cot. Moreover, θ being assumed acute (whether it is actually so or not, it does not matter), $270^{\circ} - \theta$ falls in the third quadrant, where tan is positive.

Hence,
$$\tan (270^{\circ} - \theta) = + \cot \theta$$
.

 $180^{\circ} + \theta$ has got θ associated with even multiple of 90° . Hence, the ratio does not alter in form, sec remaining sec. Also, $180^{\circ} + \theta$ falls in the third quadrant, if θ be assumed acute, where sec (by the rule "all, sin, tan, cos") is negative.

Hence,
$$\sec (180^{\circ} + \theta) = -\sec \theta$$
.

N. B. The angle ' $-\theta$ ' may be written as $0.860^{\circ} - \theta$, and 0 may be considered *even* in applying the above rule.

Thus, θ being supposed acute, $-\theta$ falls in the fourth quadrant, where cos and sec only are positive. The form of the ratio not changing in this case, $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = +\cos\theta$, etc.

30. Special angles (outside the first quadrant).

In Art. 24, putting $\theta = 60^{\circ}$, 45° , 30° and 0° respectively we can deduce the following results:

$$\sin 120^{\circ} = \sin 60^{\circ} = \frac{\sqrt{3}}{2}; \qquad \cos 120^{\circ} = -\cos 60^{\circ} = -\frac{1}{2}.$$

$$\sin 135^{\circ} = \sin 45^{\circ} = \frac{1}{\sqrt{2}}; \qquad \cos 135^{\circ} = -\cos 45^{\circ} = -\frac{1}{\sqrt{2}}.$$

$$\sin 150^{\circ} = \sin 30^{\circ} = \frac{1}{2}; \qquad \cos 150^{\circ} = -\cos 30^{\circ} = -\frac{\sqrt{3}}{2}.$$

$$\sin 180^{\circ} = \sin 0^{\circ} = 0; \qquad \cos 180^{\circ} = -\cos 0^{\circ} = -1.$$
And similarly from Arts. 27 and 28, putting $\theta = 0$,
$$\sin 270^{\circ} = -\cos 0^{\circ} = -1; \qquad \cos 270^{\circ} = \sin 0^{\circ} = 0;$$

$$\sin 360^{\circ} = \sin 0^{\circ} = 0; \qquad \cos 360^{\circ} = \cos 0^{\circ} = 1.$$

From the above we get,

 $\tan 180^{\circ} = 0$; $\tan 270^{\circ} = \pm \infty$; $\tan 360^{\circ} = 0$.

Examples worked out.

Ex. 1. Find the value of $\cot (-1575^{\circ})$.

cot
$$(-1575^{\circ})$$
 = $-\cot (1575^{\circ})$ = $-\cot (4 \times 360^{\circ} + 185^{\circ})$
= $-\cot (135^{\circ})$ = $-\cot (180^{\circ} - 45^{\circ})$
= $\cot 45^{\circ} = 1$.

Ex. 2. Find the value of cot θ - tan θ , where $\theta = \frac{17\pi}{3}$.

 $\frac{17\pi}{3} = 6\pi - \frac{\pi}{3}$, and omitting complete multiples of 360° i.e., of 2π , whereby trigonometrical ratios are not altered, we get.

$$\cot \frac{17\pi}{3} = \cot \left(-\frac{\pi}{3}\right) = -\cot \frac{\pi}{3} = -\cot 60^{\circ} = -\frac{1}{\sqrt{3}}.$$

$$\tan \frac{17\pi}{3} = \tan \left(-\frac{\pi}{3}\right) = -\tan \frac{\pi}{3} = -\tan 60^{\circ} = -\sqrt{3}.$$

$$\therefore \cot \theta - \tan \theta = -\frac{1}{\sqrt{3}} + \sqrt{3} = \frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Ex. 3. Prove that $\sin (420^{\circ}) \cos (390^{\circ}) + \cos (-300^{\circ}) \sin (-330^{\circ}) = 1.$ [H. S. 1962]

L. H. side =
$$\sin (360^{\circ} + 60^{\circ}) \cos (360^{\circ} + 30^{\circ})$$

+ $\cos (-360^{\circ} + 60^{\circ}) \sin (-360^{\circ} + 30^{\circ})$
= $\sin 60^{\circ} \cos 30^{\circ} + \cos 60^{\circ} \sin 30^{\circ}$
= $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$.

Ex. 4. Express cot (-1358°) in terms of the ratio of a positive angle less than 45°.

$$\cot (-1358^{\circ}) = \cot (-4 \times 360^{\circ} + 82^{\circ})$$
$$= \cot 82^{\circ} = \cot (90^{\circ} - 8^{\circ})$$
$$= \tan 8^{\circ}.$$

Note. Ratios of angles of any magnitude and sign can always be expressed in terms of a ratio of a positive angle less than 45°.

Ex. 5. Express

 $cos (90^{\circ} + \theta) sec (-\theta) tan (180^{\circ} - \theta)$ $sec (360^{\circ} + \theta) sin (180^{\circ} + \theta) cot (90^{\circ} - \theta)$ in its simplest form.

The given expression

$$= \frac{-\sin \theta . \sec \theta . (-\tan \theta)}{\sec \theta . (-\sin \theta) . \tan \theta}$$
$$= -1.$$

Examples IV

- 1. Write down the values of $\sin 150^{\circ}$, cot 840°, $\csc (-660^{\circ})$ and $\tan (-1125^{\circ})$.
 - 2. Find the values of $\sin\left(-\frac{11\pi}{4}\right)$, cosec $\binom{16\pi}{3}$, $\tan\left(\frac{3\pi}{2} + \frac{\pi}{3}\right)$ and $\cos\left(\frac{5\pi}{2} \frac{19\pi}{3}\right)$.
- 3. Evaluate $\sin\left(-1230^{\circ}\right) \cos\left\{\left(2n+1\right)\pi + \frac{\pi}{3}\right\}$, where n is a negative integer.
- 4. Find the value of $\sin \left\{ n\pi + (-1)^n \frac{\pi}{3} \right\}$, where n is any integer.
 - 5. Find all the values of

(i)
$$\tan \left\{ \frac{n\pi}{2} + (-1)^n \frac{\pi}{4} \right\}$$
;

(ii) cosec
$$\left\{ \frac{n\pi}{2} + (-1)^n \frac{\pi}{6} \right\}$$

where n is any integer.

6. Show that $\cos\left(2m\pi\pm\frac{\pi}{3}\right)$ and $\tan\left(m\pi+\frac{\pi}{6}\right)$ have one value each for all integral values of m.

- 7. Prove that, n being any integer
 - (i) $\cos (n\pi + a) = (-1)^n \cos a$.
 - (ii) $\tan (n\pi a) = -\tan a$.
- 8. Prove that
 - (i) $\cos \theta = -\cos (\theta 180^\circ)$.
 - (ii) $\tan \theta = -\cot (\theta \frac{3}{4}\pi)$.
- 9. Prove that
 - (i) $\sin (780^\circ) \cos (390^\circ) \sin (330^\circ) \cos (-300^\circ) = 1$.
 - (ii) $\cos 306^{\circ} + \cos 234^{\circ} + \cos 162^{\circ} + \cos 18^{\circ} = 0$.
 - (iii) $\frac{\sin 250^{\circ} + \tan 290^{\circ}}{\cot 200^{\circ} + \cos 340^{\circ}} = -1.$
- 10. Simplify

$$\sin^{3}(\pi+\theta)$$
 $\tan(2\pi-\theta)\sec^{2}(\pi-\theta)$
 $\cos^{2}(\frac{1}{2}\pi+\theta)$ $\csc^{2}\theta$ $\sin(\pi-\theta)$

and determine its value when $\theta = 225^{\circ}$.

11. Prove that

$$\sin\left(\frac{1}{2}\pi + \theta\right)\cos\left(\pi - \theta\right)\cot\left(\frac{3}{2}\pi + \theta\right)$$

$$= \sin\left(\frac{1}{2}\pi - \theta\right)\sin\left(\frac{3}{2}\pi - \theta\right)\cot\left(\frac{1}{2}\pi + \theta\right).$$

12. Evaluate

(i)
$$\sin^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \sin^2\frac{5\pi}{4} + \sin^2\frac{7\pi}{4}$$

(ii)
$$\cot \frac{\pi}{20} \cot \frac{3\pi}{20} \cot \frac{5\pi}{20} \cot \frac{7\pi}{20} \cot \frac{9\pi}{20}$$

- (iii) $\sin x + \sin (\pi + x) + \sin (2\pi + x) + \cdots$ to n terms.
- 13. If $\tan \theta = \frac{5}{18}$ and $\cos \theta$ is negative, find the value of $\frac{\sin \theta + \cos (-\theta)}{\sec (-\theta) + \tan \theta}$.
- 14. An angle θ lies between 180° and 270°, and cosec θ : $-\frac{\pi}{4}$. Find cot θ .

- 15. Express in terms of ratios of positive angles less than 45°:
 - (i) $\cot (-1054^{\circ})$.

- (ii) sin (1145°).
- (iii) sec (-1491°).
- (iv) $\cos \frac{35\pi}{9}$.
- 16. Find the values of θ , when
 - (i) $\tan \theta = -\sqrt{3}$ and θ lies between 270° and 360°.
 - (ii) $\cos \theta = -\frac{1}{2}$, and $450^{\circ} < \theta < 540^{\circ}$.
- 17. Solve for θ , giving all the possible values, when $0^{\circ} < \theta < 360^{\circ}$:
 - (i) $\cos \theta + \sqrt{3} \sin \theta = 2$.

[C. U. 1936]

- (ii) $2\sin^2\theta + 3\cos\theta = 0$.
- (iii) $3(\sec^2\theta + \tan^2\theta) = 5$.
- (iv) $\cot \theta + \tan \theta = 2 \sec \theta$.
- (v) $1-2\sin\theta-2\cos\theta+\cot\theta=0$.
- 18. If A, B, C be the angles of a triangle, show that $\sin (A+B) \cos C = \cos (A+B) + \sin C$.
- 19. If A, B, C be the angles of a triangle, show that $\frac{\tan (B+C) + \tan (C+A) + \tan (A+B)}{\tan (a-A) + \tan (2\pi-B) + \tan (3\pi-C)} = 1.$
- 20. If A, B, C, D be the angles of a quadrilateral, show that

$$\cos \frac{1}{2}(A+C) + \cos \frac{1}{2}(B+D) = 0.$$

If the quadrilateral be cyclic, then

$$\cos A + \cos B + \cos C + \cos D = 0.$$

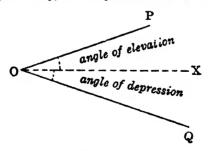
CHAPTER V

SIMPLE PRACTICAL APPLICATIONS OF TRIGONOMETRY

(Heights and Distances)

31. One of the most important applications of Trigonometry is in the determination of heights and distances of distant objects which are not directly measurable, by observations of angles subtended by those objects at the eye of the observer. These angles may be measured by instruments known as Sextants or Theodolites or by other anglemeasuring instruments. Thus, Trigonometry plays a very important part in land survey. It is also extensively used by Astronomers in determining the distances of the heavenly bodies like the sun, the moon and the stars.

Two angles are very often used in the practical applications of Trigonometry, and they are defined as follows:

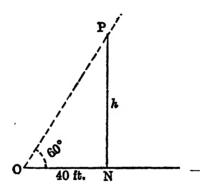


If a horizontal line OX be drawn through O, the eye of an observer, the angle which the line joining O to a point P above OX makes with OX is called the **Angle of Elevation** or altitude of P as seen from O.

If Q be below the horizontal line OX, the angle XOQ measured below OX is called the Angle of Depression of Q as seen from O.

32. Illustrative Examples.

Ex. 1. From a distance of 40 feet from the foot of a palm tree in a horizontal field, the angle of elevation of the top of the tree is observed to be 60°. Find the height of the tree.



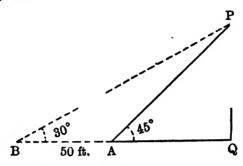
Let h ft. be the height of the tree PN, and $\angle NOP$, the angle of elevation of P as seen from O, where ON = 40 ft., is 60° .

Then,
$$\frac{h}{40}$$
 ·· tan $PON = \tan 60^{\circ} = \sqrt{3}$;

 $h = 40 \text{ s/3 ft.} = 69^{\circ}28 \cdots \text{ ft.}$

Ex. 2. From one bank of a river, the top of a building just on the opposite bank is observed to have an elevation of 45°. On receding 50 ft. from the bank, perpendicular to its edge, the angle of elevation becomes 30°. Find the breadth of the river and the height of the building.

AQ being the breadth of the river, PQ the height of the building, $\angle PAQ = 45^{\circ}$. Also, AB being 50 ft. $\angle PBQ = 30^{\circ}$.



Now,
$$\frac{BQ}{PQ} = \cot 30^{\circ}$$
, $\frac{AQ}{PQ} = \cot 45^{\circ}$.

Hence, subtracting, $\frac{AB}{PQ} = \cot 30^{\circ} - \cot 45^{\circ}$,

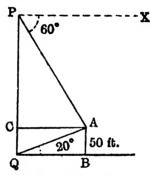
or,
$$\frac{50}{PQ} = \sqrt{3} - 1$$
;
 $\therefore PQ = \frac{50}{\sqrt{3} - 1} = \frac{50(\sqrt{3} + 1)}{2} = 68.3 \text{ ft. nearly.}$
Also, $\frac{AQ}{PQ} = \cot 45^{\circ} = 1$; $\therefore AQ = PQ = 68.3 \text{ ft.}$

Thus, the breadth of the river and the height of the building are both 68 3 ft. nearly.

Ex. 8. The angles of depression and elevation of the top of a tower 50 ft. high from the top and bottom of a second tower are 60° and 20° respectively. Find the height of the second tower to the nearest foot. [Given cot 20° = 2.747.]

PQ is the second tower, and $\angle XPA = 60^{\circ}$, $\angle BQA = 20^{\circ}$, AB = 50 ft., AC is parallel to BQ or PX, so that $\angle PAC$ = the alternate angle $XPA = 60^{\circ}$.

Now,
$$QB = \cot 20^\circ$$
, $\therefore QB = AB \cot 20^\circ$.
Also, $PC = \cot PAC = \tan 60^\circ$;
 $PC = CA \tan 60^\circ = QB \tan 60^\circ$
 $= AB \cot 20^\circ \tan 60^\circ$.



.. height
$$PQ = PC + CQ = PC + AB$$

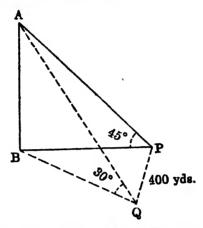
= $AB (\cot 20^{\circ} \tan 60^{\circ} + 1)$
= $50 (2.747 \times \sqrt{3} + 1)$
= $287.8... \text{ ft.} = 288 \text{ ft. nearly.}$

Ex. 4. The elevation of a hill from a place P due East of it is 45°, and at a place Q due South of P, the elevation is 30°. If the distance PQ be 400 yds., find the height of the hill.

A is the top of the hill, B is the point vertically below it on the ground. BP is due East, PQ is due South, so that BPQ is a right angle. Also ABP and ABQ are both right angles.

Now,
$$\frac{BQ}{AB} = \cot AQB = \cot 30^{\circ} = \sqrt{3}$$
,
and $\frac{BP}{AB} = \cot APB = \cot 45^{\circ} = 1$.

Hence, $BQ = AB \sqrt{3}$, BP = AB,



and
$$PQ^2 = BQ^2 - BP^2 = AB^2(3-1) = 2AB^2$$
.

$$\therefore AB = \frac{PQ}{\sqrt{2}} = \frac{PQ}{2} \cdot \sqrt{2} = 200 \sqrt{2} = 283 \text{ yds. nearly.}$$

Examples V

- 1. From the top of a tower by the seaside, 100 feet high, it was observed that the angle of depression of the bottom of a ship at anchor was 30°. Find the distance of the ship from the bottom of the tower.
- 2. Two straight roads, which cross one another, meet a river with straight course at angles 60° and 30° respectively. If it be 3 miles by the longer of the two roads, from the crossing to the river, how far is it by the shortest? If there be a foot-path which goes the shortest way from the crossing to the river, what is the distance by it?
- 3. Two poles are of equal height; a person standing midway between the line joining their bases observes the

elevation of the poles to be 30°. After walking 40 feet towards one of them, he observes that the same pole now subtends an angle of 60°. Find their height and the distance between them.

- 4. A straight palm tree 60 feet high, is broken by the wind but not completely separated, and its upper part meets the ground at an angle of 30°. Find the distance of the point where the top of the tree meets the ground, from the root, and also the height at which the tree is broken.
- 5. Two posts are 120 ft. apart, and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds the angular elevations of their tops to be complementary: Find the height of the shorter post.

 [H. S. 1961 Com.]
- 6. The Bally bridge subtends an angle of 45° at a given point at the edge of the river; 800 yds. higher up, it subtends an angle of 30°. The course of the river here is straight and perpendicular to the bridge. Find the length of the bridge.
- 7. The height of a house subtends a right angle at an opposite window, the top being 60° above a horizontal straight line through the window; find the height of the house, taking the breadth of the street to be 30 feet.
- 8. From an aeroplane vertically over a straight road, the angles of depression of two consecutive milestones are observed to be 45° and 60°; find the height of the aeroplane.
- 9. From a ship sailing due South-East at the rate of 5 miles an hour, a light-house is observed to be 30° North of East, and after 4 hours, it is seen due North; find the distance of the light-house from the final position of the ship.
- 10. The shadow of a tower standing on a level plane is found to be 40 feet longer when the sun's altitude is 45° than when it is 60°. Find the height of the tower.

- 11. From the lower window of a house the angular elevation of a church-steeple is found to be 45° and from a window 20 feet above, the elevation is 30°. How far is the church from the house?
- 12. A light-house facing East sends out a fan-shaped beam of light extending from S. E. to N. E. An observer sailing due North, after meeting the light continues to see it for $10\sqrt{2}$ minutes. When leaving the fan of light, the ship is 10 miles from the light-house. Find the speed of the ship.
- 13. A pole 100 ft. high stands vertically at the centre of a horizontal equilateral triangle, each side of which subtends an angle of 60° at the top of the pole. Find the side of the triangle.
- 14. Two chimneys are of equal height. A person standing between them in the line joining their bases observes the elevation of the nearer one to be 60°. After walking 80 feet in a direction at right angles to the line joining their bases, he observes the elevations of the two to be 45° and 30° respectively. Find the height and the distance between them.
- 15. At the foot of a mountain the elevation of its summit is 45° ; after ascending 1 mile towards the mountain up an incline of 30° , the elevation changes to 60° . Find the height of the mountain.
- 16. From a station, two light-houses A and B are seen in directions North and 30° East of North respectively; if A were one-third as far off as it really is, it would appear due West of B. If the distance of B from the station be 10 miles, find the distance of B from A.
- 17. A person walking along a straight road observes a tall tree standing in front of a tower, both being on the road before him. The elevation of the top of the tower is 45°, and that of the top of the tree 80°; on advancing 100 feet he finds the tower and the tree to have the same elevation 60°; supposing the height of the eye of the man to be 5 feet, find the height of the tower and of the tree.

- 18. A man on the top of a rock rising on a seashore, observes a boat coming towards it at an angle of depression 30°; 10 minutes later the angle of depression is 60°. The height of the rock being 4000 feet, find the speed of the boat in miles per hour.
- 19. A person walking along a straight level road observes the elevation of the top of a hill to be 60° when he is nearest the hill, and after walking 200 yards in a direction perpendicular to the direction of the hill from this point, observes the elevation to be 30°. Find the approximate height of the hill.
- 20. A square tower stands on a horizontal plane. From a point in this plane, only three of its upper corners are visible, and their angles of elevation are 45°, 60°, 45°. Find the ratio of the height of the tower to its breadth.
- 21. Two wheels, the sum of whose radii is 10 feet, are placed flatly on a table with their centres at a distance of 20 ft. An endless string, quite stretched, is partly wrapped round the wheels and crosses itself between them. Show that the length of the string is nearly 76'5 feet.
- 22. On a still day, from a station A an airship is observed due North at an elevation of 60° , while from a station B it is observed due East at an elevation of 45° . At this instant of observation, a parachute message is dropped from the airship, and the observer at A has to walk a mile to reach the message. Find the distance between the two stations.
- 23. From the foot of a column the angle of elevation of the top of a tower is 45° and from the top of the column the angle of depression of the bottom of the tower is 30°. A man walks 10 ft. from the bottom of the column towards the tower and notices the angle of elevation of its top to be 60°. Find the height of the column.

CHAPTER VI

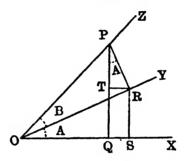
COMPOUND ANGLES

33. To Prove that

$$\sin (A+B) = \sin A \cos B + \cos A \sin B$$

 $\cos (A+B) = \cos A \cos B - \sin A \sin B$

when A and B are positive and acute and $(A+B) < 90^{\circ}$.



Let a revolving line starting from the position OX trace out an angle XOY = A and then revolving further, trace out an angle YOZ = B; then $\angle XOZ = A + B$.

In OZ, the bounding line of the compound angle A+B, take any point P and draw PQ and PR perpendiculars to OX and OY respectively; also draw RS and RT perpendiculars to OX and PQ respectively.

From the right-angled $\triangle POQ$,

$$\sin (A+B) = \frac{PQ}{OP} = \frac{QT+TP}{OP} = \frac{RS+PT}{OF} = \frac{RS}{OP} + \frac{PT}{OP}$$

$$= \frac{RS}{OR} \cdot \frac{OR}{OP} + \frac{PT}{PR} \cdot \frac{PR}{OP}$$

$$= \sin A \cos B + \cos TPR \sin B.$$

Now,
$$\angle TPR = 90^{\circ} - \angle TRP = \angle TRO = \angle ROS = A$$
.

$$\therefore$$
 sin $(A+B) = \sin A \cos B + \cos A \sin B$. Again,

$$\cos (A+B) = \frac{OQ}{OP} = \frac{OS - QS}{OP} = \frac{OS - TR}{OP} = \frac{OS}{OP} - \frac{TR}{OP}$$

$$= \frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP}$$

$$= \cos A \cos B - \sin A \sin B$$

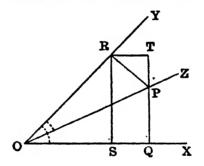
$$= \cos A \cos B - \sin A \sin B.$$

34. To prove that

$$\sin (A-B) = \sin A \cos B - \cos A \sin B$$

 $\cos (A-B) = \cos A \cos B + \sin A \sin B$,

when A and B are positive and acute, and A > B.



Let a revolving line start from the position OX and trace out an angle XOY = A and then revolving back trace out an angle YOZ = B; then $\angle XOZ = A - B$.

In OZ, the bounding line of the compound angle A-B, take any point P, and draw PQ and PR perpendiculars to OX and OY respectively; and draw RS and RT perpendiculars to OX and QP produced respectively.

From the right-angled $\triangle POQ$,

$$\sin (A - B) = \frac{PQ}{OP} = \frac{TQ - PT}{OP} = \frac{RS - PT}{OP} = \frac{RS}{OP} - \frac{PT}{OP}$$

$$=\frac{RS}{OR},\frac{OR}{OP}-\frac{PT}{PR},\frac{PR}{OP}$$

 $= \sin A \cos B - \cos TPR \cdot \sin B$.

But $\angle TPR - 90^{\circ} - \angle TRP = \angle YRT = \angle YOX = A$.

 $\therefore \sin (A - B) = \sin A \cos B - \cos A \sin B.$

Again,

$$\cos (A - B) = \frac{OQ}{OP} = \frac{OS + SQ}{OP} = \frac{OS + RT}{OP} = \frac{OS}{OP} + \frac{RT}{OP}$$
$$= \frac{OS}{OR} \cdot \frac{OR}{OP} + \frac{RT}{RP} \cdot \frac{RP}{OP}$$
$$= \cos A \cos B + \sin TPR \cdot \sin B$$
$$= \cos A \cos B + \sin A \sin B.$$

Obs. In the above Geometrical proofs, it is assumed that the angles A, B, A+B are all less than a right angle and that A-B is positive. If the angles are not so restricted, the same method of proof (there being some modifications in the figures) will apply, due attention being paid to the signs of the quantities involved.*

Thus, the above formulæ are perfectly general.

Note 1. The sum or difference of two or more angles is called a Compound angle; such as A+B, A-B, A+B+C etc.

The expansions $\sin (A \pm B)$ and $\cos (A \pm B)$ are generally called the "Addition formulæ or Addition and Subtraction Theorems".

Note 2. Assuming the truth of the above formulæ for acute angles, they can be shown to be true for angles of any magnitude, as follows:

Let us consider $\sin (A + B)$.

Let A and B be soute and $A+B < 90^{\circ}$.

Let $A_1 = 90^{\circ} + A$; $B_1 = B$.

Now,
$$\sin (A_1 + B_1) = \sin \{(90^\circ + A) + B\} = \sin \{90^\circ + (A + B)\}$$

= $\cos (A + B) = \cos A \cos B - \sin A \sin B$ [by Art. 33]
= $\sin (90^\circ + A) \cos B + \cos (90^\circ + A) \sin B$
= $\sin A_1 \cos B_1 + \cos A_1 \sin B_1$.

^{*}See Appendix, Arts. 2-4. Also for alternative proofs of Arts. 38 and 34, see Appendix, Art. 5.

Again, let
$$A_2 = -A$$
, $B_3 = B$.
Then, $\sin (A_2 + B_2) = \sin (-A + B) = -\sin (A - B)$
 $= -\sin A \cos B + \cos A \sin B$, [by Art. 34]
 $= \sin (-A) \cos B + \cos (-A) \sin B$
 $= \sin A_2 \cos B_2 + \cos A_3 \sin B_3$.

Thus, the above formulæ remain true if any of the two angles is either increased by 90°, or has its sign changed.

In the same way it may be shown that the other three formulæ for $\cos(A+B)$, $\sin(A-B)$ and $\cos(A-B)$ will continue to hold good unchanged in form, if any of the two angles be either increased by 90° or has its sign changed.

Now starting from positive acute-angled values of A and B, combining the two processes of increasing one of the angles by 90°, and reversing the sign of any one, we can arrive at values of A and B of any magnitude, positive or negative and the four formulæ will still hold good.

Thus, the formulæ for $\sin (A \pm B)$ and $\cos (A \pm B)$ are perfectly general.

sin 75° = sin (45° + 30°) = sin 45° cos 30° + cos 45° sin 30°
=
$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$
.

$$\cos 75^{\circ} = \cos (45^{\circ} + 30^{\circ}) = \cos 45^{\circ} \cos 30^{\circ} - \sin 45^{\circ} \sin 30^{\circ}$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

 $\sin 15^{\circ} \pm \sin (45^{\circ} - 30^{\circ}) = \sin 45^{\circ} \cos 30^{\circ} - \cos 45^{\circ} \sin 30^{\circ}$ and $\cos 15^{\circ} = \cos (45^{\circ} - 30^{\circ}) = \cos 45^{\circ} \cos 30^{\circ} + \sin 45^{\circ} \sin 30^{\circ}$ therefore, substituting the values of $\sin 45^{\circ}$, $\cos 45^{\circ}$ etc. as before, we get

$$\sin 15^\circ = \frac{\sqrt{3-1}}{2\sqrt{2}}$$
 and $\cos 15^\circ = \frac{\sqrt{3+1}}{2\sqrt{2}}$.

Note. The values of sin 15° and cos 15° can also be deduced from the fact that

$$\sin 15^\circ = \sin (90^\circ - 75^\circ) = \cos 75^\circ$$

and $\cos 15^\circ = \cos (90^\circ - 75^\circ) = \sin 75^\circ$.

Ex. 2. Show that

(i)
$$\sin (A+B) \sin (A-B) = \sin^2 A - \sin^2 B$$

= $\cos^2 B - \cos^2 A$.

(ii)
$$\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B$$

= $\cos^2 B - \sin^2 A$.

(i) Left side

$$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$$

$$=\sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$

$$=\sin^2 A (1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$$

$$=\sin^2 A - \sin^2 B$$

$$=(1-\cos^2 A)-(1-\cos^2 B)=\cos^2 B-\cos^2 A.$$

(ii) Left side

$$=(\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$$

$$=\cos^2 A \cos^2 B - \sin^2 A \sin^2 B$$

$$=\cos^2 A (1-\sin^2 B)-(1-\cos^2 A)\sin^2 B$$

$$=\cos^2 A - \sin^2 B$$

$$= (1 - \sin^2 A) - (1 - \cos^2 B) = \cos^2 B - \sin^2 A.$$

Note. The results of Ex. 1 and Ex. 2 are very useful and should be carefully remembered.

36. To prove that

(i)
$$\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii)
$$\tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

We have

$$\tan (A+B) = \frac{\sin (A+B)}{\cos (A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Now, dividing the numerator and denominator by $\cos A \cos B$, we have,

$$\tan (A+B) = \frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}$$

$$\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}$$

$$\cos A \cos B + \cos A \cos B$$

$$\tan A + \tan B$$

$$1 - \tan A \tan B$$

Again,

$$\tan (A-B) = \frac{\sin (A-B)}{\cos (A-B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

Now, dividing the numerator and denominator by $\cos A \cos B$, we have, as before,

$$\tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}.$$

37. To prove that

(i)
$$\cot (A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

(ii)
$$\cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

(i)
$$\cot (A+B) = \frac{\cos (A+B)}{\sin (A+B)} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$$

Now, dividing the numerator and denominator by $\sin A \sin B$, we have,

$$\cot (A+B) = \frac{\frac{\cos A \cos B}{\sin A \sin B} - \frac{\sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B}{\sin A \sin B} + \frac{\cos A \sin B}{\sin A \sin B}}$$
$$= \frac{\cot A \cot B - 1}{\cot B + \cot A}.$$

(ii)
$$\cot (A-B) = \frac{\cos (A-B)}{\sin (A-B)} = \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}$$

Now, dividing the numerator and denominator by $\sin A \sin B$, we have, as before,

$$\cot (A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

38. Ex. 1. Find the values of tan 75° and tan 15°.

$$\tan 75^{\circ} = \tan (45^{\circ} + 30^{\circ}) = \frac{\tan 45^{\circ} + \tan 30^{\circ}}{1 - \tan 45^{\circ} \tan 30^{\circ}}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)(\sqrt{3} + 1)}{3 - 1}$$

$$= \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}.$$

$$\tan 15^{\circ} = \tan (45^{\circ} - 30^{\circ}) = \frac{\tan 45^{\circ} - \tan 30^{\circ}}{1 + \tan 45^{\circ} \tan 30^{\circ}}$$
$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$
$$= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

Ex. 2. Show that

(i)
$$\tan (45^{\circ} + A) = \frac{1 + \tan A}{1 - \tan A}$$

(ii)
$$tan(45^{\circ} - A) = \frac{1 - tan A}{1 + tan A}$$

(i) Left side =
$$\frac{\tan 45^{\circ} + \tan A}{1 - \tan 45^{\circ} \tan A} = \frac{1 + \tan A}{1 - \tan A}$$
.

(ii) This result follows similarly.

$$\cot 2A + \tan A = \csc 2A$$
.

[C. U. 1947]

Left side =
$$\frac{\cos 2A}{\sin 2A} + \frac{\sin A}{\cos A} = \frac{\cos 2A \cos A + \sin 2A \sin A}{\sin 2A \cos A}$$

= $\frac{\cos (2A - A)}{\sin 2A \cos A} = \frac{\cos A}{\sin 2A \cos A} = \frac{1}{\sin 2A}$
= $\frac{\cos 2A}{\sin 2A}$

39. To find the expansions of

(i)
$$\sin (A+B+C)$$

(ii)
$$\cos(A+B+C)$$

(iii)
$$tan(A+B+C)$$
.

(i)
$$\sin (A+B+C)$$

 $= \sin \{(A+B)+C\}$
 $= \sin (A+B) \cos C + \cos (A+B) \sin C$
 $= (\sin A \cos B + \cos A \sin B) \cos C$
 $+ (\cos A \cos B - \sin A \sin B) \sin C$
 $= \sin A \cos B \cos C + \sin B \cos C \cos A$

Note 1. The expansion of $\sin (A+B+C)$ can be easily put in the form

 $\cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C).$

 $+\sin C\cos A\cos B - \sin A\sin B\sin C$

(ii)
$$\cos (A+B+C)$$

 $=\cos\{(A+B)+C\}$
 $=\cos (A+B)\cos C-\sin (A+B)\sin C$
 $=(\cos A\cos B-\sin A\sin B)\cos C$
 $-(\sin A\cos B+\cos A\sin B)\sin C$
 $=\cos A\cos B\cos C-\cos A\sin B\sin C$
 $=\cos B\sin C\sin A-\cos C\sin A\sin B$

Note 2. The expansion of $\cos(A+B+C)$ can be easily put in the form

cos A cos B cos C (1-tan B tan C-tan C tan A-tan A tan B).

(iii)
$$\tan (A+B+C)$$

$$= \tan \{(A+B)+C\}$$

$$= \tan (A+B)+\tan C$$

$$= 1-\tan (A+B)\tan C$$

$$= \frac{1-\tan A \tan B}{1-\tan A \tan B} + \tan C$$

$$= \frac{1-\tan A + \tan B}{1-\tan A \tan B} + \tan C$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1-\tan B \tan C - \tan A \tan B}$$

Note 3 The expansion of $t_{in}(A+B+C)$ can also be obtained thus,

$$\tan (A+B+C) = \frac{\sin (A+B+C)}{\cos (A+B+C)}.$$

Now, write down the expansions of $\sin (A+B+C)$ and $\cos (A+B+C)$ and divide the numerator and denominator by $\cos A \cos B \cos C$ or simply write down the expansions of $\sin (A+B+C)$ and $\cos (A+B+C)$ as given in Notes 1 and 2.

Obs. Formulæ for the Trigonometrical functions of the sum of four, five or more angles can be similarly obtained.

Examples VI

Show that (Ex. 1 to 20):—

1. (i) $\sin (A - B) = \frac{16}{68}$ and $\cos (A + B) = \frac{83}{68}$, if A and B are acute and if $\sin A = \frac{8}{10}$, $\cos B = \frac{12}{10}$.

(ii) $\cos 68^{\circ} 20' \cos 8^{\circ} 20' + \cos 81^{\circ} 40' \cos 21^{\circ} 40' = \frac{1}{2}$.

(iii)
$$\sec (x-y) = \frac{8}{8}\frac{5}{4}$$
, if $\sec x = \frac{17}{8}$, cosec $y = \frac{5}{4}$.

2. (i)
$$\sin A \sin (B-C) + \sin B \sin (C-A) + \sin C \sin (A-B) = 0$$
.

(ii)
$$\cos A \sin (B-C) + \cos B \sin (C-A) + \cos C \sin (A-B) = 0.$$

(iii)
$$\sin (B+C) \sin (B-C) + \sin (C+A) \sin (C-A) + \sin (A+B) \sin (A-B) = 0.$$

(iv)
$$\sin (\alpha - \theta) \sin (\beta - \gamma) + \sin (\beta - \theta) \sin (\gamma - \alpha) + \sin (\gamma - \theta) \sin (\alpha - \beta) = 0.$$

- 3. $\cos (60^{\circ} A) \cos (30^{\circ} B) \sin (60^{\circ} A) \sin (30^{\circ} B)$ = $\sin (A + B)$.
- 4. (i) $\sin (n+1) x \cos (n-1)x \cos (n+1) x \sin (n-1) x$ = $\sin 2x$.
 - (ii) $\sin 2\theta \cos \theta + \cos 2\theta \sin \theta$ = $\sin 4\theta \cos \theta - \cos 4\theta \sin \theta$.
- 5. $\frac{\sin B}{\sin A} = \frac{\sin (2A+B)}{\sin A} 2\cos (A+B)$.
- 6. $\frac{\sin (B-C)}{\cos B \cos C} + \frac{\sin (C-A)}{\cos C \cos A} + \frac{\sin (A-B)}{\cos A \cos B} = 0.$
- 7. $\frac{\sin{(B-C)}}{\sin{B}\sin{C}} + \frac{\sin{(C-A)}}{\sin{C}\sin{A}} + \frac{\sin{(A-B)}}{\sin{A}\sin{B}} = 0.$
- 8. $\tan (A+B) \tan (A-B) = \frac{\sin^2 A \sin^2 B}{\cos^2 A \sin^2 B}$
- 9. $\tan^2 A \tan^2 B = \frac{\sin (A+B) \sin (A-B)}{\cos^2 A \cos^2 B}$
- 10. (i) $\frac{\tan (\alpha + \beta) \tan \alpha}{1 + \tan (\alpha + \beta) \tan \alpha} = \tan \beta.$
- (ii) If $A+B+C=\pi$ and $\cos A=\cos B\cos C$, show that $\tan A=\tan B+\tan C$. [C. U. 1942]
 - 11. $1 + \tan 2\theta \tan \theta = \sec 2\theta$.
 - 12. $\cot \theta \cot 2\theta = \csc 2\theta$.
 - 13. $\tan 20^{\circ} + \tan 25^{\circ} + \tan 25^{\circ} \tan 20^{\circ} = 1$.
 - 14. (i) $\tan (45^{\circ} + A) = \frac{\cos A + \sin A}{\cos A \sin A}$
 - (ii) $\sqrt{2} \sin (45^{\circ} + A) = \sin A + \cos A$.
 - 15. $\frac{\cos 8^{\circ} + \sin 8^{\circ}}{\cos 8^{\circ} \sin 8^{\circ}} = \tan 53^{\circ}$.

16.
$$\tan (45^{\circ} + A) \tan (45^{\circ} - A) = 1$$
.

17.
$$\tan (A+B) + \tan (A-B) = \frac{\sin 2A}{\cos^2 A - \sin^2 B}$$

18.
$$\frac{\sin (x+y)}{\sin (x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}$$

19.
$$\cot (45^{\circ} + x) = \frac{\cot x - 1}{\cot x + 1} = \frac{\cos x - \sin x}{\cos x + \sin x}$$

$$20. \quad \sec (x+y) = \frac{\sec x \sec y}{1-\tan x \tan y}.$$

- 21. Find the expansions of $\sin (A-B+C)$ and $\tan (A-B-C)$.
- 22. Express cot (A+B+C) in terms of cot A, cot B, cot C.
- 23. (i) If $a \cos (x + a) = b \cos (x a)$, prove that $(a + b) \tan x = (a b) \cot a$.
 - (ii) If $\sin a \sin \beta \cos a \cos \beta + 1 = 0$, show that $1 + \cot a \tan \beta = 0$. [C. U. 1939]
 - (iii) If $A+B+C=\pi$ and $\cos A=\cos B\cos C$, then $\cot B\cot C=\frac{1}{2}$.
- 24. If $\tan \theta = \frac{a \sin x + b \sin y}{a \cos x + b \cos y}$ then $a \sin (\theta - x) + b \sin (\theta - y) = 0$.
- 25. An angle θ is divided into two parts α , β such that $\tan \alpha$: $\tan \beta = x : y$; prove that

$$\sin (\alpha - \beta) = \frac{x - y}{x + y} \sin \theta.$$

26. If $\cos (\beta - \gamma) + \cos (\gamma - a) + \cos (a - \beta) = -\frac{3}{2}$, show that $\Sigma \cos a = 0$ and $\Sigma \sin a = 0$.

-CHAPTER VII

TRANSFORMATION OF PRODUCTS AND SUMS

40. Transformation of products into sums or differences.

We have from Arts. 33 and 34,

$$\sin A \cos B + \cos A \sin B = \sin (A + B) \quad \cdots \quad (1)$$

$$\sin A \cos B - \cos A \sin B = \sin (A - R). \quad \cdots \quad (2)$$

Adding (1) and (2), we get

$$2 \sin A \cos B = \sin (A+B) + \sin (A-B). \quad \cdots \quad (3)$$

Subtracting (2) from (1), we get

2 cos A sin
$$B = \sin (A + B) - \sin (A - B)$$
. ... (4)

Again, from Arts. 33 and 34, we have,

$$\cos A \cos B - \sin A \sin B = \cos (A + B) \quad \cdots \quad (5)$$

$$\cos A \cos B + \sin A \sin B = \cos (A - B). \quad \cdots \quad (6)$$

Adding (5) and (6), we get

$$2\cos A\cos B = \cos (A+B) + \cos (A-B)$$
. ... (7)

Subtracting (5) from (6), we get

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B). \quad \cdots \quad (8)$$

Thus, we have the following formula for transforming a product of two sines and cosines into the sum or the difference of two sines or two cosines.

$$2 \sin A \cos B = \sin (A+B) + \sin (A-B). \quad \cdots \quad (I)$$

$$2 \cos A \sin B = \sin (A+B) - \sin (A-B)$$
. ... (II)

$$2 \cos A \cos B = \cos (A+B) + \cos (A-B)$$
. ... (III)

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B). \quad \cdots \quad (IV)$$

41. Transformation of sums or differences into products.

Let
$$A+B=C$$
, and $A-B=D$,
then $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$.

Making these substitutions for A and B in the results (3), (4), (7), (8) of Art. 40 and noting that the relation (8) can be written as

$$\cos (A+B) - \cos (A-B) = -2 \sin A \sin B$$

$$= 2 \sin A \sin (-B),$$

we have the following four formulæ for transforming the sum or the difference of two sines only or two cosines only into a product of sines and cosines.

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \cdots$$
 (I)

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \cdots (II)$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \cdots (III)$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2} \cdots$$
 (IV)

Note. The following concise verbal statement of the above four formulæ is sometimes very convenient.

- (i) sine + sine = 2 sin (sum). cos (dif.),
- (ii) sine sine = 2 cos (sum). sin (diff.).
- (iii) $\cos + \cos = 2 \cos (\frac{1}{2} \text{ sum}) \cdot \cos (\frac{1}{2} \text{ diff.})$.
- (iv) $\cos -\cos = 2 \sin (\frac{1}{2} \operatorname{sum})$. $\sin (\frac{1}{2} \operatorname{diff. reversed})$.
- 42. Ex. 1. Prove that
 - (i) $\cos 20^{\circ} \cos 40^{\circ} \cos 80^{\circ} = \frac{1}{8}$.
 - (ii) $\cos 80^{\circ} + \cos 40^{\circ} \cos 20^{\circ} = 0$.

(i) Left side =
$$\frac{1}{2}$$
.cos 20° (2 cos 40° cos 80°)
= $\frac{1}{2}$ cos 20° (cos 120° + cos 40°)
= $\frac{1}{2}$ cos 20° ($-\frac{1}{2}$ + cos 40°)
= $-\frac{1}{4}$ cos 20° + $\frac{1}{3}$ cos 20° cos 40°
= $-\frac{1}{4}$ cos 20° + $\frac{1}{4}$ (cos 60° + cos 20°)
= $-\frac{1}{4}$ cos 20° + $\frac{1}{4}$ ($\frac{1}{2}$ + cos 20°)
= $\frac{1}{8}$.

(ii) Left side =
$$(\cos 80^{\circ} + \cos 40^{\circ}) - \cos 20^{\circ}$$

= $2 \cos 60^{\circ} \cos 20^{\circ} - \cos 20^{\circ}$
= $2.\frac{1}{2} \cos 20^{\circ} - \cos 20^{\circ} = 0$.

Ex. 2. Show that

$$\frac{\sin \theta + \sin 2\theta + \sin 4\theta + \sin 5\theta}{\cos \theta + \cos 2\theta + \cos 4\theta + \cos 5\theta} = \tan 3\theta.$$
Numerator = $(\sin 5\theta + \sin \theta) + (\sin 4\theta + \sin 2\theta)$

= $2 \sin 3\theta \cos 2\theta + 2 \sin 3\theta \cos \theta$ = $2 \sin 3\theta (\cos 2\theta + \cos \theta)$:

Denominator = $(\cos 5\theta + \cos \theta) + (\cos 4\theta + \cos 2\theta)$ = $2 \cos 3\theta \cos 2\theta + 2 \cos 3\theta \cos \theta$ = $2 \cos 3\theta (\cos 2\theta + \cos \theta)$.

left side =
$$\frac{2 \sin 3\theta (\cos 2\theta + \cos \theta)}{2 \cos 3\theta (\cos 2\theta + \cos \theta)} = \frac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta$$
.

Ex. 3. Express 4 cos A cos B cos C as the sum of four cosines.

Ex. 4. Express as the product of three sines $\sin (B+C-A)+\sin (C+A-B)+\sin (A+B-C)$ $-\sin (A+B+C)$.

Grouping together the first two terms and grouping together the last two terms, the given expression

$$= 2 \sin C \cos (B-A) + 2 \cos (A+B) \sin (-C)$$

$$= 2 \sin C \{\cos (B-A) - \cos (A+B)\}$$

$$=2 \sin C (2 \sin B \sin A)$$

$$=4 \sin A \sin B \sin C$$
.

Examples VII

Prove that (Ex. 1 to 17):-

1.
$$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{A + B}{2} \cot \frac{A - B}{2}$$

2.
$$\frac{\cos A + \cos B}{\cos B - \cos A} = \cot \frac{A+B}{2} \cot \frac{A-B}{2}.$$

3.
$$\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$$

4.
$$\sin \theta \sin (60^{\circ} - \theta) \sin (60^{\circ} + \theta) = \frac{1}{4} \sin 3\theta$$
.

5.
$$\cos \theta \cos (60^{\circ} - \theta) \cos (60^{\circ} + \theta) = \frac{1}{4} \cos 3\theta$$
.

6.
$$(\sin 3\alpha + \sin \alpha) \sin \alpha + (\cos 3\alpha - \cos \alpha) \cos \alpha = 0$$
.

7.
$$\cos (A-D) \sin (B-C) + \cos (B-D) \sin (C-A) + \cos (C-D) \sin (A-B) = 0.$$

8.
$$\cos 20^{\circ} \cos 40^{\circ} \cos 60^{\circ} \cos 80^{\circ} = \frac{1}{18}$$
.

9.
$$\sin 20^{\circ} \sin 40^{\circ} \sin 60^{\circ} \sin 80^{\circ} = \frac{8}{18}$$
.

10.
$$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A + B}{2}$$

11.
$$\frac{\sin A - \sin B}{\cos B - \cos A} = \cot \frac{A + B}{2}$$

12.
$$\frac{\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta.$$

18.
$$\frac{\sin 2A + \sin 5A - \sin A}{\cos 2A + \cos 5A + \cos A} = \tan 2A$$
.

14.
$$\frac{\sin{(\alpha+\beta)}-2\sin{\alpha}+\sin{(\alpha-\beta)}}{\cos{(\alpha+\beta)}-2\cos{\alpha}+\cos{(\alpha-\beta)}}=\tan{\alpha}.$$

15.
$$\frac{\cos 7a + \cos 3a - \cos 5a - \cos a}{\sin 7a - \sin 3a - \sin 5a + \sin a} = \cot 2a$$
.

16.
$$\sin 2A + \sin 2B + \sin 2C - \sin 2(A + B + C)$$

= $4 \sin (B + C) \sin (C + A) \sin (A + B)$.

17.
$$\cos A + \cos B + \cos C + \cos (A + B + C)$$

= $4 \cos \frac{B+C}{2} \cos \frac{C+A}{2} \cos \frac{A+B}{2}$

18. If $\sin x = k \sin y$, prove that

$$\tan \frac{1}{2}(x-y) = \frac{k-1}{k+1} \tan \frac{1}{2}(x+y).$$

- 19. If $\cos x + \cos y = \frac{1}{3}$ and $\sin x + \sin y = \frac{1}{4}$, prove that $\tan \frac{1}{2}(x+y) = \frac{3}{4}$.
- 20. If $x \cos a + y \sin a = k = x \cos \beta + y \sin \beta$, prove that

$$\frac{x}{\cos \frac{1}{2}(\alpha+\beta)} = \frac{y}{\sin \frac{1}{2}(\alpha+\beta)} = \cos \frac{1}{2}(\alpha-\beta)$$

21. If $\sin \theta + \sin \phi = a$, $\cos \theta + \cos \phi = b$, prove that $\tan \frac{\theta - \phi}{2} = \pm \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}$.

22. Prove that
$$\frac{\cos 10^{\circ} - \sin 10^{\circ}}{\cos 10^{\circ} + \sin 10^{\circ}} = \tan 35^{\circ}$$
.

[Note that $\sin \theta = \cos (90^{\circ} - \theta)$ and $\cos \theta = \sin (90^{\circ} \pm \theta)$.]

23. If cosec
$$A + \sec A = \csc B + \sec B$$
, then Law $A = \csc \frac{1}{2}(A + B)$
 $(A + \tan B) = \cot \frac{1}{2}(A + B)$. [P. U. 1936]

24. Prove that

$$\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = 2 \cot^n \frac{A - B}{2},$$

or zero, according as n is even or odd.

[P. U. 1938]

CHAPTER VIII

MULTIPLE ANGLES

43. Trigonometrical ratios of angle 2A.

From Art. 33, we have,

$$\sin (A+B) = \sin A \cos B + \cos A \sin B$$
,
 $\cos (A+B) = \cos A \cos B - \sin A \sin B$.

Putting
$$B = A$$
, in the first formula, we get $\sin 2A = \sin A \cos A + \cos A \sin A$

$$= 2 \sin A \cos A. \qquad \cdots (1)$$

Putting B = A, in the second formula, we get $\cos 2A = \cos A \cdot \cos A - \sin A \cdot \sin A$

$$= \cos^2 A - \sin^2 A \qquad \cdots \qquad (2)$$

$$=(1-\sin^2 A)-\sin^2 A=1-2\sin^2 A$$
 ... (3)

and also =
$$\cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1$$
. ... (4)

By Art. 36, $\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

Putting B=A, in the above formula, we get

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \cdot \tan A} = \frac{2 \tan A}{1 - \tan^2 A} \cdot \cdots (5)$$

Similarly, putting B=A in the value of cot (A+B) as given in Art. 37, we get cot $2A = \frac{\cot^2 A - 1}{2 \cot A}$ (6)

From formulæ (3) and (4), we obtain, by transposition,

$$1 + \cos 2A = 2 \cos^2 A \qquad \cdots \qquad \cdots \qquad (7)$$

$$1-\cos 2A-2\sin^2 A.\qquad \cdots \qquad \cdots \qquad (8)$$

By division,
$$\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$$
. ... (9)

We may also note that

$$1 + \sin 2A = \cos^2 A + \sin^2 A + 2 \sin A \cos A$$

$$= (\cos A + \sin A)^2$$

$$1 - \sin 2A = \cos^2 A + \sin^2 A - 2 \sin A \cos A$$

$$= (\cos A - \sin A)^2.$$

Note. Since the addition formulæ are perfectly general (i.e., true for all values of A and B), the above formulæ, being deduced from addition formulæ, are also perfectly general.

44. Trigonometrical ratios of angle 3A.

$$\sin 3A = \sin (A + 2A) = \sin A \cos 2A + \cos A \sin 2A$$

= $\sin A (1 - 2 \sin^2 A) + \cos A \cdot 2 \sin A \cos A$
[By Art. 43]
= $\sin A (1 - 2 \sin^2 A) + 2 \sin A (1 - \sin^2 A)$.

 \therefore sin 3A = 3 sin A - 4 sin 3A .

$$\cos 3A = \cos (A + 2A) = \cos A \cos 2A - \sin A \sin 2A$$

$$= \cos A (2 \cos^2 A - 1) - \sin A \cdot 2 \sin A \cos A$$

$$= \cos A (2 \cos^2 A - 1) - 2 \cos A \cdot \sin^2 A$$

$$= 2 \cos^3 A - \cos A - 2 \cos A \cdot (1 - \cos^2 A).$$

 $\therefore \cos 3A = 4 \cos^3 A - 3 \cos A.$

$$\tan 3A = \tan (A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \frac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A \cdot \frac{2 \tan A}{1 - \tan^2 A}}$$

$$= \tan A \cdot \frac{1 - \tan^2 A}{1 - \tan^2 A}$$

$$= \tan A \cdot \frac{(1 - \tan^2 A) + 2 \tan A}{(1 - \tan^2 A) - 2 \tan^2 A}$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

Obs. By a method similar to that of the previous article the trigonometrical ratios of any higher multiple of A can be expressed in terms of those of A.

45. Ex. 1. Express sin 2A and cos 2A in terms of tan A.

$$\sin 2A = 2 \sin A \cos A = 2 \frac{\sin A}{\cos A} \cdot \cos^2 A$$

$$= 2 \tan A \frac{1}{\sec^2 A} = \frac{2 \tan A}{1 + \tan^2 A}.$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - \cos^2 A \cdot \frac{\sin^2 A}{\cos^2 A}$$

$$= \cos^2 A \left(1 - \frac{\sin^2 A}{\cos^2 A}\right) = \frac{1}{\sec^2 A} (1 - \tan^2 A)$$

$$= \frac{1 - \tan^2 A}{1 + \cos^2 A}.$$

Ex. 2. Express cos 4A in terms of cos A.

Putting
$$\theta = 2A$$
, $\cos 4A = \cos 2\theta = 2 \cos^2 \theta - 1$
= $2 (\cos 2A)^2 - 1$
= $2 (2 \cos^2 A - 1)^2 - 1$
= $8 \cos^4 A - 8 \cos^2 A + 1$.

Ex. 3. Show that $\frac{1 - \tan^2 (45^{\circ} - A)}{1 + \tan^2 (45^{\circ} - A)} = \sin 2A$.

Let
$$\theta = 45^{\circ} - A$$
; then

Left side =
$$\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^3 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta}$$
$$= \cos^2 \theta - \sin^2 \theta = \cos 2\theta$$
$$= \cos (90^\circ - 2A) = \sin 2A.$$

Examples VIII

Prove the following identities (Ex. 1 to 24):—

1.
$$\frac{\sin 2A}{1+\cos 2A} = \tan A.$$

$$2. \quad \frac{\sin 2A}{1-\cos 2A} = \cot A.$$

3.
$$\cot A - \tan A = 2 \cot 2A$$
.

4. (i)
$$(2 \cos \theta + 1)(2 \cos \theta - 1) = 2 \cos 2\theta + 1$$
.

(ii)
$$\tan \theta (1 + \sec 2\theta) = \tan 2\theta$$
.

5.
$$\frac{\cot A - \tan A}{\cot A + \tan A} = \cos 2A.$$

6.
$$\tan A + \cot A = 2 \csc 2A$$
.

7.
$$\cos^4\theta - \sin^4\theta = \cos 2\theta.$$

8.
$$\cos^6 \theta - \sin^6 \theta = \cos 2\theta (1 - \frac{1}{4} \sin^2 2\theta)$$
.

9.
$$\cos^6 \theta + \sin^6 \theta = \frac{1}{4} (1 + 3 \cos^2 2\theta)$$
.

10.
$$\frac{\sin^2 a - \sin^2 \beta}{\sin a \cos a - \sin \beta \cos \beta} = \tan (a + \beta).$$

11. (i)
$$\frac{1-\cos 2\theta + \sin 2\theta}{1+\cos 2\theta + \sin 2\theta} = \tan \theta$$
. [C. U. 1938]

(ii)
$$\frac{\sin a - \sqrt{1 + \sin 2a}}{\cos a - \sqrt{1 + \sin 2a}} = \cot a$$
. [a being positive and

acute, and the square root being taken with positive sign.]

12.
$$\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} - \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = 2 \tan 2\theta.$$

13. (i)
$$\frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$$
.

(ii)
$$\frac{\sin 4\theta}{\cos 2\theta} \cdot \frac{1 - \cos 2\theta}{1 - \cos 4\theta} = \tan \theta.$$

14. (i)
$$\frac{\cos A - \sin A}{\cos A + \sin A} = \sec 2A - \tan 2A$$
.

(ii)
$$\frac{\cos^3\theta + \sin^3\theta}{\cos\theta + \sin\theta} = 1 - \frac{1}{2}\sin 2\theta.$$

15.
$$\cos^3 A \cos 3A + \sin^3 A \sin 3A = \cos^3 2A$$
.

16. (i)
$$4(\cos^8 10^\circ + \sin^8 20^\circ) = 3(\cos 10^\circ + \sin 20^\circ)$$
.

(ii)
$$\cos 10^{\circ} - \sqrt{3} \sin 10^{\circ} = 2 \sin 20^{\circ}$$
.

17.
$$\tan 3\theta - \tan 2\theta - \tan \theta = \tan 3\theta \tan 2\theta \tan \theta$$
.

18.
$$\cot A = \cot A = \tan A = 1.$$

19.
$$\frac{1}{\tan 3\theta - \tan \theta} - \frac{1}{\cot 3\theta - \cot \theta} = \cot 2\theta.$$

20.
$$\sin 8\theta = 8 \sin \theta \cos \theta \cos 2\theta \cos 4\theta$$
.

21. (i)
$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$
.

(ii)
$$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$
.

22. (i)
$$\cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}$$
.

(ii)
$$\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^3 \theta + \tan^3 \theta}$$

23. (i)
$$\cos (120^{\circ} - A) + \cos A + \cos (120^{\circ} + A) = 0$$
.

(ii)
$$\cos^2(A-120^\circ)+\cos^2A+\cos^2(A+120^\circ)=\frac{3}{2}$$
.

24.
$$\frac{\tan 2^n \theta}{\tan \theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta) \cdots (1 + \sec 2^n \theta).$$

25. If
$$\alpha = \frac{3}{7}\pi$$
, show that

(i)
$$\cos a \cos 2a \cos 4a = \frac{1}{8}$$
.

(ii)
$$\cos a + \cos 2a + \cos 4a = -\frac{1}{2}$$
.

(iii)
$$\sin \alpha + \sin 2\alpha + \sin 4\alpha = \frac{\sqrt{7}}{8}$$

26. If
$$\theta = \frac{\pi}{2^n + 1}$$
, prove that

$$2^n \cos \theta \cos 2\theta \cos 2^2\theta \cdots \cos 2^{n-1}\theta = 1$$

- 27. (i) If $\tan x = b/a$, find the value of $a \cos 2x + b \sin 2x$.
- (ii) If $\tan^2 x + 2 \tan x \tan 2y = \tan^2 y + 2 \tan y \tan 2x$, prove that each side = 1, or, else, $\tan x = \pm \tan y$.
 - 28. If $\tan^2 \theta = 1 + 2 \tan^2 \phi$, show that $\cos 2\phi = 1 + 2 \cos 2\theta$.
 - 29. (i) If 2 tan a=3 tan β , prove that

$$\tan (\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}.$$
 [C. U. 1946]

(ii) If $\frac{\tan (\alpha - \beta - \gamma)}{\tan (\alpha - \beta + \gamma)} = \frac{\tan \gamma}{\tan \beta}$, show that either. $\sin (\beta - \gamma) = 0$, or, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$.

- 30. If α and β are acute angles and $\cos 2\alpha = \frac{3\cos 2\beta 1}{3 \cos 2\beta}$, show that $\tan \alpha = \sqrt{2} \tan \beta$. [C. U. 1941]
 - 31. If $\cos \theta = \frac{1}{2} (a + a^{-1})$, show that
 - (i) $\cos 2\theta = \frac{1}{2} (a^2 + a^{-2}).$
 - (ii) $\cos 3\theta = \frac{1}{2} (a^8 + a^{-8}).$

Show that (Ex. 32 to 36):-

- 32. $\sin^4\theta = \frac{3}{8} \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$.
- 33. $\cos^8 \theta + \sin^8 \theta = 1 \sin^2 2\theta + \frac{1}{8} \sin^4 2\theta$.
- 34. $\tan \left(\frac{\pi}{4} + A\right) + \tan \left(\frac{\pi}{4} A\right) = 2 \sec 2A$.
- 35. $\cos^3\theta \frac{\sin 3\theta}{3} + \sin^3\theta \frac{\cos 3\theta}{3} = \frac{\sin 4\theta}{4}$
- $36. \quad \cos 4x \cos 4y$
 - $= 8 (\cos x \cos y)(\cos x + \cos y)(\cos x \sin y)$ $\times (\cos x + \sin y).$

CHAPTER IX

SUBMULTIPLE ANGLES

46. From the usual formulæ for multiple angles, namely

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$1 + \cos 2A = 2 \cos^2 A ; 1 - \cos 2A = 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

putting $A = \frac{1}{2}\theta$ and $\frac{1}{3}\theta$ respectively we derive the following formulæ for submultiple angles:

$$\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$$

$$\cos \theta = \cos^{2}\frac{1}{2}\theta - \sin^{2}\frac{1}{2}\theta = 2 \cos^{2}\frac{1}{2}\theta - 1 = 1 - 2 \sin^{2}\frac{1}{2}\theta$$

$$1 + \cos \theta = 2 \cos^{2}\frac{1}{2}\theta ; 1 - \cos \theta = 2 \sin^{2}\frac{1}{2}\theta$$

$$\tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^{2}\frac{1}{2}\theta}$$

$$\sin \theta = 3 \sin \frac{1}{3}\theta - 4 \sin^{3}\frac{1}{3}\theta$$

$$\cos \theta = 4 \cos^{3}\frac{1}{3}\theta - 3 \cos \frac{1}{3}\theta$$

$$\tan \theta = \frac{3 \tan \frac{1}{3}\theta - \tan^{3}\frac{1}{3}\theta}{1 - 3 \tan^{2}\frac{1}{2}\theta}.$$

47. Values of $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$ in terms of $\cos \theta$.

From $\cos \theta = 2 \cos^2 \frac{1}{2}\theta - 1 = 1 - 2 \sin^2 \frac{1}{2}\theta$, we at once deduce

$$\sin \frac{1}{2}\theta = \pm \sqrt{\frac{1}{2}(1 - \cos \theta)}$$

$$\cos \frac{1}{2}\theta = \pm \sqrt{\frac{1}{2}(1 + \cos \theta)}.$$

48. Ambiguity of signs explained.

When $\cos \theta$ is given and not θ , θ and consequently $\frac{1}{2}\theta$ has a series of values as will be explained in Chapter XI. Thus, $\frac{1}{2}\theta$ may lie in any quadrant and $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$ will then have corresponding signs.

If the quadrant in which $\frac{1}{2}\theta$ lies be known, for example, when θ is given along with $\cos \theta$, there is no ambiguity in choosing the proper signs of $\cos \frac{1}{2}\theta$ and $\sin \frac{1}{2}\theta$, as shown in the following example.

Ex. Find $\sin 22\frac{1}{2}$ ° and $\cos 22\frac{1}{2}$ °.

$$\sin 22\frac{1}{2}^{\circ} = + \sqrt{\frac{1}{2}}(1 - \cos 45^{\circ}) = \sqrt{\frac{1}{2}}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{1}{2}\sqrt{2 - \sqrt{2}}$$

$$\cos 22\frac{1}{2}^{\circ} = + \sqrt{\frac{1}{2}}(1 + \cos 45^{\circ}) = \sqrt{\frac{1}{2}}\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{1}{2}\sqrt{2 + \sqrt{2}}.$$

49. Values of $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$ in terms of $\sin \theta$.

We know that
$$\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$$

and $1 = \cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta$.

Therefore,
$$1 + \sin \theta = (\cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta)^2$$
,
and $1 - \sin \theta = (\cos \frac{1}{2}\theta - \sin \frac{1}{2}\theta)^2$.

Hence,
$$\cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta = \pm \sqrt{1 + \sin \theta}$$

 $\cos \frac{1}{2}\theta - \sin \frac{1}{2}\theta = \pm \sqrt{1 - \sin \theta}$.

Thus,
$$\cos \frac{1}{2}\theta = \pm \frac{1}{2}\sqrt{1+\sin \theta} \pm \frac{1}{2}\sqrt{1-\sin \theta}$$

and $\sin \frac{1}{2}\theta = \pm \frac{1}{2}\sqrt{1+\sin \theta} \mp \frac{1}{2}\sqrt{1-\sin \theta}$.

. 50. Ambiguity of signs explained.

As before, when $\sin \theta$ is given, and not θ , θ has a series of values for the given value of $\sin \theta$ as will be explained in Chapter XI; $\frac{1}{2}\theta$ may therefore lie in any one of two possible quadrants.

$$\cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta = \sqrt{2} \sin \left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)$$

and
$$\cos \frac{1}{2}\theta - \sin \frac{1}{2}\theta = \sqrt{2} \sin (\frac{1}{2}\pi - \frac{1}{2}\theta)$$

will have their signs determined accordingly.

Thus, $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$ will be definitely known.

Ex. Find sin 15° and cos 15°.

We have,
$$\cos 15^{\circ} + \sin 15^{\circ} = + \sqrt{1 + \sin 30^{\circ}} = \sqrt{1 + \frac{1}{2}}$$

 $\cos 15^{\circ} - \sin 15^{\circ} = + \sqrt{1 - \sin 30^{\circ}} = \sqrt{1 - \frac{1}{2}}$.

[$\cos 15^{\circ} - \sin 15^{\circ} = \sqrt{2} \sin (\frac{1}{4}\pi - 15^{\circ})$ and is clearly positive.]

Thus,
$$\cos 15^{\circ} = \frac{1}{2} \left(\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}} \right) = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$\sin 15^\circ = \frac{1}{2} \left(\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}} \right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

51. $\tan \frac{1}{2}\theta$ in terms of $\tan \theta$.

From the formula, $\tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta}$

i.e., $\tan \theta \tan^2 \frac{1}{2}\theta + 2 \tan \frac{1}{2}\theta - \tan \theta = 0$,

we easily deduce

$$\tan \frac{1}{2}\theta = \frac{-1 + \sqrt{1 + \tan^2 \theta}}{\tan \theta}.$$

The reason of the ambiguity is similar to those of the previous cases.

52. Ratios of $\frac{1}{3}\theta$ from those of θ .

By solving the cubic equation

$$\sin \theta = 3 \sin \frac{1}{3}\theta - 4 \sin^3 \frac{1}{3}\theta \qquad \cdots \qquad \cdots \qquad \cdots \qquad (1)$$

we get $\sin \frac{1}{2}\theta$, if $\sin \theta$ be known.

Similarly, by solving the cubic equations $\cos \theta = 4 \cos^3 \theta - 8 \cos \frac{1}{2}\theta \cdots$ (2)

and
$$\tan \theta = \frac{3 \tan \frac{1}{3}\theta - \tan^3 \frac{1}{3}\theta}{1 - 3 \tan^3 \frac{1}{3}\theta}$$
 ... (3)

we derive values of $\cos \frac{1}{2}\theta$ from those of $\cos \theta$, and of $\tan \frac{1}{2}\theta$ from those of $\tan \theta$ respectively.

53. Ratios of 18° and 36°.

Let
$$\theta = 18^{\circ}$$
; then $5\theta = 90^{\circ}$; $\therefore 2\theta = 90^{\circ} - 3\theta$.

$$\therefore \sin 2\theta = \cos 3\theta, \text{ or, } 2 \sin \theta \cos \theta = \cos \theta (4 \cos^2 \theta - 3).$$

As $\cos \theta$ (i.e. $\cos 18^{\circ}$) is not zero, we have $2 \sin \theta = 4 \cos^2 \theta - 3 = 1 - 4 \sin^2 \theta$.

$$4 \sin^2 \theta + 2 \sin \theta - 1 = 0$$

$$\therefore \sin \theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{(\pm \sqrt{5}-1)}{4}.$$

Now, as θ here is a positive acute angle, therefore, rejecting the negative value, we get

sin 18° =
$$\frac{1}{4}(\sqrt{5} - 1)$$
.
cos 18° = $+\sqrt{1-\sin^2 18}$ ° = $\frac{1}{4}(\sqrt{10+2}\sqrt{5})$.
cos 36° = $1-2\sin^2 18$ ° = $\frac{1}{4}(\sqrt{5} + 1)$.
sin 36° = $\sqrt{1-\cos^2 36}$ ° = $\frac{1}{4}(\sqrt{10-2}\sqrt{5})$.

Note. Since 54° and 36° are complementary and 72° and 18° are complementary, from the above values we easily get the trigonometrical ratios of 54° and 72°.

54. Ratios of 3° and multiples of 3°.

$$\sin 3^{\circ} = \sin (18^{\circ} - 15^{\circ}) = \sin 18^{\circ} \cos 15^{\circ} - \cos 18^{\circ} \sin 15^{\circ}$$

= $\frac{1}{16} (\sqrt{5} - 1)(\sqrt{6} + \sqrt{2}) - \frac{1}{8} (\sqrt{3} - 1)(\sqrt{5} + \sqrt{5}),$

on substituting the values of sin 18°, cos 15°, etc.

Similarly,

$$\cos 3^{\circ} = \frac{1}{5} (\sqrt{3} + 1)(\sqrt{5} + \sqrt{5}) + \frac{1}{15}(\sqrt{6} - \sqrt{2})(\sqrt{5} - 1).$$

From a knowledge of the ratios of 3°, 15°, 18°, 30°, 36° and 45°, we can deduce the ratios for all angles which are multiples of 3°, (for, $6^{\circ} = 36^{\circ} - 30^{\circ}$; $9^{\circ} = 45^{\circ} - 36^{\circ}$;

 $12^{\circ} = 30^{\circ} - 18^{\circ}$: $21^{\circ} = 36^{\circ} - 15^{\circ}$; etc.). For angles greater than 45°, the ratios may be deduced from those of their complements which are less than 45°.

Ex. Show that

$$\sin x = 2^n \cos \frac{x}{2} \cos \frac{x}{2^2} \cos \frac{x}{2^3} \cdots \cos \frac{x}{2^n} \sin \frac{x}{2^n}.$$

We have,
$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$\sin\frac{x}{2} - 2\sin\frac{x}{2^2}\cos\frac{x}{2^2}$$

$$\sin\frac{x}{2^2} = 2\sin\frac{x}{2^8}\cos\frac{x}{2^8}$$

Similarly, $\sin \frac{x}{\Omega^{n-1}} = 2 \sin \frac{x}{\Omega^n} \cos \frac{x}{\Omega^n}$

Hence, $\sin x = 2^n \cos \frac{x}{2} \cos \frac{x}{2^n} \cos \frac{x}{2^n} \cdot \dots \cdot \cos \frac{x}{2^n} \sin \frac{x}{2^n}$

Examples IX

Prove that (Ex. 1 to 14):

1.
$$\frac{1-\cos A}{\sin A} = \tan \frac{A}{2}$$
 2.
$$\frac{1+\cos A}{\sin A} = \cot \frac{A}{2}$$

$$2. \quad \frac{1+\cos A}{\sin A} = \cot \frac{A}{2}$$

3.
$$\left(\sin \frac{A}{2} \pm \cos \frac{A}{2}\right)^2 = 1 \pm \sin A$$
.

4. sec
$$\theta$$
 + tan θ = tan $(\frac{1}{2}\pi + \frac{1}{2}\theta)$.

[C. U. 1939]

5. (i)
$$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\frac{\theta}{2}$$
.

(ii)
$$\frac{\sin \frac{1}{2}a - \sqrt{1 + \sin a}}{\cos \frac{1}{2}a - \sqrt{1 + \sin a}} = \cot \frac{a}{2} \text{ where } 0 < a < \pi,$$

and the square root is taken with positive sign.

6. (i)
$$\frac{1+\sin x}{1-\sin x} = \tan^2 \left(\frac{\pi}{2} + \frac{x}{2}\right)$$
.

(ii)
$$\frac{2 \sin \theta - \sin 2\theta}{2 \sin \theta + \sin 2\theta} = \tan^{2} \frac{1}{2}\theta.$$

7. (i)
$$\frac{1+\tan\frac{1}{2}A}{1-\tan\frac{1}{2}A} = \frac{1+\sin A}{\cos A}$$
.

(ii) cot $\beta = \frac{1}{2} \left(\cot \frac{1}{2}\beta - \tan \frac{1}{2}\beta \right)$.

8. (i)
$$\frac{\sin 2\theta}{1+\cos 2\theta}$$
, $\frac{\cos \theta}{1+\cos \theta} = \tan \frac{\theta}{2}$.

(ii) $8 \sin^4 \frac{1}{2}\theta - 8 \sin^2 \frac{1}{2}\theta + 1 = \cos 2\theta$.

9.
$$\sin \theta = \frac{2 \tan \frac{1}{2} \theta}{1 + \tan^2 \frac{1}{2} \theta}$$
 10. $\cos \theta = \frac{1 - \tan^2 \frac{1}{2} \theta}{1 + \tan^2 \frac{1}{2} \theta}$

- 11: $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 4 \cos^2 \frac{1}{2} (x y)$.
- 12. $\tan 6^{\circ} \tan 42^{\circ} \tan 66^{\circ} \tan 78^{\circ} = 1$.
- 13. $\tan 7\frac{1}{3}^{\circ} = \sqrt{6} \sqrt{3} + \sqrt{2} 2$.
- 14. $2 \cos \frac{1}{16}\pi = \sqrt{2 + \sqrt{2} + \sqrt{2}}$

15. (i) If
$$\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \cdot \tan \frac{\phi}{2}$$
, show that
$$\cos \phi = \frac{\cos \theta - e}{1-e \cos \theta}.$$

- (ii) If $\tan \theta = \frac{\sin \alpha \sin \beta}{\cos \alpha + \cos \beta}$ show that one of the values of $\tan \frac{1}{2}\theta$ is $\tan \frac{1}{2}a \tan \frac{1}{2}\beta$.
- 16. If $\sin a + \sin \beta = a$ and $\cos a + \cos \beta = b$, find the value of $\cos (a + \beta)$.
- 17. (i) Prove that $2 \sin \frac{1}{2}A = \pm \sqrt{1 + \sin A} \pm \sqrt{1 \sin A}$, and determine which are the correct signs when $270^{\circ} > A > 180^{\circ}$. [B. H. U. I., 1931]
 - (ii) If $\theta = 240^{\circ}$, is the following statement correct? $2 \sin \frac{1}{2}\theta = \sqrt{1 + \sin \theta} - \sqrt{1 - \sin \theta}$.

If not, how must it be modified?

18. If $A = 320^{\circ}$, prove that

$$\tan\frac{A}{2} = \frac{-1 + \sqrt{1 + \tan^2 A}}{\tan A}.$$

CHAPTER X

TRIGONOMETRICAL IDENTITIES

55. Many interesting identities involving functions of three or more angles can be established when there exists a relation among the angles. The most important of these identities are those in which the three angles are connected by the relation that their sum is equal to two right angles. In establishing this latter kind of identities, it will be necessary to make frequent use of the properties of supplementary and complementary angles.

Thus, since
$$A+B+C=\pi$$
,

$$\therefore B+C=\pi-A.$$

$$\therefore \quad \sin (B+C) = \sin (\pi - A) = \sin A.$$

Similarly,
$$\sin (C+A) = \sin B$$
; $\sin (A+B) = \sin C$.

Again,
$$\cos (B+C) = \cos (\pi - A) = -\cos A$$
.

Similarly,
$$\cos (C+A) = -\cos B$$
; $\cos (A+B) = -\cos C$.
 $\tan (B+C) = \tan (\pi - A) = -\tan A$.

Similarly, $\tan (C + A) = -\tan B$; $\tan (A + B) = -\tan C$.

Again, since,
$$\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$$
,

$$\therefore \sin\left(\frac{B}{2} + \frac{C}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{A}{2}\right) = \cos$$
Similarly, $\sin\left(\frac{C}{2} + \frac{A}{2}\right) = \cos\frac{B}{2}$;
 $\sin\left(\frac{A}{2} + \frac{B}{2}\right) = \cos\frac{C}{2}$.
Again, $\cos\left(\frac{B}{2} + \frac{C}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin$

Similarly,
$$\cos\left(\frac{C}{2} + \frac{A}{2}\right) = \sin\frac{B}{2}$$
;
$$\cos\left(\frac{A}{2} + \frac{B}{2}\right) = \sin\frac{C}{2}.$$

$$\tan\left(\frac{B}{2} + \frac{C}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{A}{2}\right) = \cot\frac{A}{2}.$$
Similarly, $\tan\left(\frac{C}{2} + \frac{A}{2}\right) = \cot\frac{B}{2}$;
$$\tan\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\frac{C}{2}.$$
56. Ex. 1. If $A + B + C = \pi$, prove that
$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$
[C. U. 1931, '33, '35; H. S. '61, Comp.]
Left side = $(\sin 2A + \sin 2B) + \sin 2C$

$$= 2 \sin (A + B) \cos (A - B) + 2 \sin C \cos C$$

$$= 2 \sin C \cos (A - B) + 2 \sin C \cos C$$

$$= 2 \sin C \left[\cos(A - B) + \cos C\right]$$

$$= 2 \sin C \left[\cos(A - B) - \cos(A + B)\right]$$

$$= 2 \sin C \cdot 2 \sin A \sin B$$

$$= 4 \sin A \sin B \sin C.$$
Ex. 2. If $A + B + C = \pi$, prove that
$$\cos 2A + \cos 2B + \cos 2C = -4 \cos A \cos B \cos C - 1.$$
Left side = $(\cos 2A + \cos 2B) + \cos 2C$

$$= 2 \cos (A + B) \cos (A - B) + 2 \cos^2 C - 1$$
Left side = $(\cos 2A + \cos 2B) + \cos 2C$

Left side =
$$(\cos 2A + \cos 2B) + \cos 2C$$

= $2 \cos (A + B) \cos (A - B) + 2 \cos^2 C - 1$
= $-2 \cos C \cos (A - B) + 2 \cos^2 C - 1$
[: $A + B + C = \pi$.]
= $-2 \cos C [\cos (A - B) - \cos C] - 1$
= $-2 \cos C [\cos (A - B) + \cos (A + B)] - 1$
[: $A + B + C = \pi$.]
= $-2 \cos C . 2 \cos A \cos B - 1$

 $=-4\cos A\cos B\cos C-1$.

Ex. 3. If $A+B+C=\pi$, prove that

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

[C. U. 1910, '29; H. S. '60, Comp.]

Left side =
$$(\sin A + \sin B) + \sin C$$

= $2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2}$
= $2 \cos \frac{C}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2}$
[$\therefore \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$]
= $2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} + \sin \frac{C}{2} \right]$
= $2 \cos \frac{C}{2} \left[\cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right]$
[$\therefore \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$]
= $2 \cos \frac{C}{2} \cdot 2 \cos \frac{A}{2} \cos \frac{B}{2}$
= $4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

Ex. 4. If $A+B+C=\pi$, prove that

 $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Left side =
$$(\cos A + \cos B) + \cos C$$

= $2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + 1 - 2 \sin^2 \frac{C}{2}$
= $2 \sin \frac{C}{2} \cos \frac{A-B}{2} - 2 \sin^2 \frac{C}{2} + 1$
 $\left[\because \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \cdot \right]$

$$= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \sin \frac{C}{2} \right] + 1$$

$$= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] + 1$$

$$\left[\because \frac{A}{2} + \frac{B}{2} + \frac{C}{2} \right]$$

$$= 2 \sin \frac{C}{2} \cdot 2 \sin \frac{A}{2} \sin \frac{B}{2} + 1$$

$$= 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

Ex. 5. If $A+B+C=\pi$, prove that tan A+tan B+tan C=tan A tan B tan C. [H. S. 1961]

Since, $B+C=\pi-A$,

$$\therefore \tan (B+C) = \tan (\pi - A).$$

$$\frac{\tan B + \tan C}{1 - \tan B \tan C} = -\tan A,$$

i.e.,
$$\tan B + \tan C = -\tan A (1 - \tan B \tan C)$$

= $-\tan A + \tan A \tan B \tan C$.

:.
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$
.

Otherwise:

$$\tan (A+B+C)=\tan \pi=0.$$

$$\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B} = 0.$$

Since, the fraction is zero, numerator must be zero.

:.
$$tan A + tan B + tan C - tan A tan B tan C = 0$$
,

i.e.,
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$
.

Ex. 6. If
$$A+B+C=\pi$$
, prove that

$$\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1.$$

[C. U. 1936, '39]

Since,
$$A + B + C = \pi$$
, $\therefore \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$.
 $\therefore \tan\left(\frac{B}{2} + \frac{C}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{A}{2}\right)$.
 $\therefore \frac{\tan\frac{B}{2} + \tan\frac{C}{2}}{1 - \tan\frac{B}{2}\tan\frac{C}{2}} = \cot\frac{A}{2} = \frac{1}{\tan\frac{A}{2}}$,
or, $\tan\frac{A}{2}\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = 1 - \tan\frac{B}{2}\tan\frac{C}{2}$.

On simplification, the required result follows.

Otherwise:

$$\frac{1/\tan\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right) = \cot\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right) = \cot\frac{\pi}{2}}{1 - \tan\frac{B}{2}\tan\frac{C}{2} - \tan\frac{C}{2}\tan\frac{A}{2} - \tan\frac{A}{2}\tan\frac{B}{2}} = 0.$$

Now the value of the fraction being zero, its numerator must be zero.

$$\therefore 1 - \tan \frac{B}{2} \tan \frac{C}{2} - \tan \frac{C}{2} \tan \frac{A}{2} - \tan \frac{A}{2} \tan \frac{B}{2} = 0,$$

whence the required result follows.

Ex. 7. If
$$A+B+C=\pi$$
, prove that
$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$$
Right side = $2 \cos \frac{\pi - A}{4} \left[2 \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4} \right]$

$$= 2 \cos \frac{\pi - A}{4} \left[\cos \frac{2\pi - (B+C)}{4} + \cos \frac{B-C}{4} \right]$$

$$= 2 \cos \frac{\pi - A}{4} \left[\cos \frac{\pi + A}{4} + \cos \frac{B-C}{4} \right]$$
[: $2\pi - (B+C) = \pi + \pi - (B+C) = \pi + A$, since $A+B+C=\pi$.]

$$= 2 \cos \frac{\pi - A}{4} \cos \frac{\pi + A}{4} + 2 \cos \frac{\pi - A}{4} \cos \frac{B - C}{4}$$

$$= \left(\cos \frac{\pi}{2} + \cos \frac{A}{2}\right) + 2 \cos \frac{B + C}{4} \cos \frac{B - C}{4}$$

$$[\because A + B + C = \pi.]$$

$$= \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}.$$

Note. Since, $\cos \frac{1}{4} (\pi - A) = \sin \left(\frac{1}{4} \pi - \frac{1}{4} (\pi - A) \right) = \sin \frac{1}{4} (\pi + A)$ and $\cos \frac{1}{4} (\pi - A) = \cos \frac{1}{4} (A + B + C - A) = \cos \frac{1}{4} (B + C)$.

... We have also,
$$\cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C$$

= $4 \sin \frac{1}{4} (\pi + A) \sin \frac{1}{4} (\pi + B) \sin \frac{1}{4} (\pi + C)$
= $4 \cos \frac{1}{4} (B + C) \cos \frac{1}{4} (C + A) \cos \frac{1}{4} (A + B)$.

Ex. 8. If
$$A+B+C=\pi$$
, prove that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$$
.

[C. U. 1932, '37, '47]

$$\cos^{2}A + \cos^{2}B + \cos^{2}C$$

$$= \frac{1}{2}(2 \cos^{2}A + 2 \cos^{2}B) + \cos^{2}C$$

$$= \frac{1}{2}(1 + \cos 2A + 1 + \cos 2B) + \cos^{2}C$$

$$= 1 + \frac{1}{2}(\cos 2A + \cos 2B) + \cos^{2}C$$

$$= 1 + \cos (A + B) \cos (A - B) + \cos C \cos C$$

$$= 1 - \cos C \cos (A - B) - \cos C \cos (A + B)$$

$$[\because A + B = \pi - C.]$$

$$= 1 - \cos C [\cos (A - B) + \cos (A + B)]$$

$$= 1 - \cos C [2 \cos A \cos B]$$

$$= 1 - 2 \cos A \cos B \cos C.$$

whence the required result follows.

Ex. 9. Show that

$$tan (\beta - \gamma) + tan (\gamma - \alpha) + tan (\alpha - \beta)$$

$$= tan (\beta - \gamma) tan (\gamma - \alpha) tan (\alpha - \beta).$$

Let
$$A = \beta - \gamma$$
, $B = \gamma - \alpha$, $C = \alpha - \beta$;
then $A + B + C = \beta - \gamma + \gamma - \alpha + \alpha - \beta = 0$.

$$A = \tan (A + B + C) = \tan 0 = 0$$
.

$$\therefore$$
 tan $A + \tan B + \tan C = \tan A \tan B \tan C$.

Now, substituting the values for A, B, C, the required result follows.

Ex. 10. If
$$x + y + z = xyz$$
, prove that $x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) = 4xyz$.

Putting $x = \tan \alpha$, $y = \tan \beta$, $z = \tan \gamma$, in the given relation, we have

 $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$.

... by transposition,

$$\tan \alpha (1 - \tan \beta \tan \gamma) = -(\tan \beta + \tan \gamma)$$

i.e.,
$$\tan \alpha = -\frac{\tan \beta + \tan \gamma}{1 - \tan \beta \tan \gamma} = -\tan (\beta + \gamma)$$
.

$$\therefore \quad a = \pi - (\beta + \gamma), \quad \therefore \quad a + \beta + \gamma = \pi, \quad \therefore \quad 2a + 2\beta + 2\gamma = 2\pi.$$

$$\therefore \tan (2a + 2\beta + 2\gamma) = \tan 2\pi = 0.$$

Therefore, as in Ex. 5 above,

 $\tan 2\alpha + \tan 2\beta + \tan 2\gamma = \tan 2\alpha \tan 2\beta \tan 2\gamma$.

Now, expressing $\tan 2a$, $\tan 2\beta$, $\tan 2\gamma$ in terms of $\tan a$, $\tan \beta$, $\tan \gamma$ and substituting x, y, z, for them, we get,

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

On simplification, the required result follows.

Examples X

If
$$A+B+C=\pi$$
, prove that $(Ex. 1 \text{ to } 16):$

$$\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$
.

[H. S. 1962]

2. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.

3.
$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

4.
$$\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$$
.

5.
$$(\cot B + \cot C)(\cot C + \cot A)(\cot A + \cot B)$$

= cosec A cosec B cosec C.

6.
$$\frac{\cot B + \cot C}{\tan B + \tan C} + \frac{\cot C + \cot A}{\tan C + \tan A} + \frac{\cot A + \cot B}{\tan A + \tan B} = 1.$$

7.
$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}$$

 $= 1 + 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4}$
 $= 1 + 4 \sin \frac{B + C}{4} \sin \frac{C + A}{4} \sin \frac{A + B}{4}$

8.
$$\cos^2 2A + \cos^2 2B + \cos^2 2C$$

= 1 + 2 \cos 2A \cos 2B \cos 2C.

9.
$$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$$
.

10.
$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

11.
$$\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2.$$

[C. U. 1949]

12.
$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

13.
$$\sin (B+C-A) + \sin (C+A-B) + \sin (A+B-C)$$

= 4 $\sin A \sin B \sin C$

$$\sin (B+2O) + \sin (C+2A) + \sin (A+2B)$$

$$= 4 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}.$$

15.
$$\cos^2 A + \cos^2 B + 2 \cos A \cos B \cos C = \sin^2 C$$
.

16.
$$\cos \frac{A}{2} \cos \frac{B-C}{2} + \cos \frac{B}{2} \cos \frac{C-A}{2} + \cos \frac{C}{2} \cos \frac{A-B}{2}$$

 $=\sin A + \sin B + \sin C$.

- 17. If $\alpha + \beta + \gamma = \frac{1}{2}\pi$, prove that
 - (i) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma = 1$.

[C. U. 1943]

- (ii) $\tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1$.
- 18. If A, B, C, D are the angles of a quadrilateral, prove that
 - (i) $\tan A + \tan B + \tan C + \tan D$ $\cot A + \cot B + \cot C + \cot D$ $= \tan A \tan B \tan C \tan D$
 - (ii) $\cos A + \cos B + \cos C + \cos D$ = $4 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (B + C) \cos \frac{1}{2} (C + A)$.
 - 19. Show that
 - (i) $\cos^2 (\beta \gamma) + \cos^2 (\gamma a) + \cos^2 (a \beta)$ = 1 + 2 $\cos (\beta - \gamma) \cos (\gamma - a) \cos (a - \beta)$.
 - (ii) $\sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta \cos (\alpha + \beta) = \sin^2 (\alpha + \beta)$.
- (iii) $\cos^2\theta + \cos^2(a+\theta) 2 \cos a \cos \theta \cos (a+\theta)$ is independent of θ .
 - 20. (i) If $a + \beta = \gamma$, show that $\cos^2 a + \cos^2 \beta + \cos^2 \gamma = 1 + 2 \cos a \cos \beta \cos \gamma$.

 [C. U. 1940]
 - (ii) If $\alpha + \beta + \gamma = 2\pi$, show that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma 2 \cos \alpha \cos \beta \cos \gamma = 1$.
- 21. If $\cos (A+B) \sin (C+D) = \cos (A-B) \sin (C-D)$, show that

 $\cot A \cot B \cot C = \cot D.$

- 22. If A+B+C=2S, prove that
 - (i) $\sin (S-A) + \sin (S-B) + \sin (S-C) \sin S$ = $4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.
 - (ii) $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C 1$ = 4 cos S cos (S - A) cos (S - B) cos (S - C).

- 23. If $A+B+C=n\pi$ (n being zero or an integer), then $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
- 24. Show that, if $a + \beta + \gamma = \pi$.

$$\tan (\beta + \gamma - a) + \tan (\gamma + a - \beta) + \tan (\alpha + \beta - \gamma)$$

$$= \tan (\beta + \gamma - a) \tan (\gamma + a - \beta) \tan (\alpha + \beta - \gamma).$$

- 25. If $A+B+C=\pi$, prove that
 - (i) $\sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B = \sin A \sin B \sin C$.
 - (ii) $\cos A \sin B \sin C + \cos B \sin C \sin A + \cos C \sin A \sin B = 1 + \cos A \cos B \cos C$.
 - (iii) $\sin 5A + \sin 5B + \sin 5C$

$$=4\cos\frac{5A}{2}\cos\frac{5B}{2}\cos\frac{5C}{2}$$

- (iv) $(\tan A + \tan B + \tan C)(\cot A + \cot B + \cot C)$ = $1 + \sec A \sec B \sec C$.
- 26. If $\cos A + \cos B + \cos C = 0$, show that $\cos 3A + \cos 3B + \cos 3C = 12 \cos A \cos B \cos C.$ [Write $\cos 3A = 4 \cos^2 A 3 \cos A$, etc.]
- 27. If $x + y + z = \frac{1}{2}\pi$, prove that $\cos (x y z) + \cos (y z x) + \cos (z x y)$ $-4 \cos x \cos y \cos z = 0.$
- 28. Show that

$$\sin (y - z) + \sin (z - x) + \sin (x - y) + 4 \sin \frac{y - z}{2} \sin \frac{z - x}{2} \sin \frac{x - y}{2} = 0.$$

- 29. If x + y + z = 0, show that $\cot (z + x y) \cot (x + y z) + \cot (x + y z) \cot (y + z x) + \cot (y + z x) \cot (z + x y) = 1$.
- 30. If x+y+z=xyz, prove that $\frac{3x-x^3}{1-3x^3} + \frac{3y-y^3}{1-3y^2} + \frac{3z-z^3}{1-3z^2} = \frac{3x-x^3}{1-3x^3} \cdot \frac{3y-y^3}{1-3y^2} \cdot \frac{3z-z^3}{1-3z^3}$

CHAPTER XI

TRIGONOMETRICAL EQUATIONS AND GENERAL VALUES

57. It will be apparent from Chapter IV that there are infinitely many angles, the trigonometrical ratios of which have a given value. For example, if $\sin\theta = \frac{1}{2}$, one value of θ (the smallest positive value) is known to be 30°. Now, sines of supplementary angles are equal. Hence, $\sin 150^\circ$ being equal to $\sin 30^\circ$ is also $\frac{1}{2}$. Again, angles differing from 30° or 150° by complete multiples of 360° will have their sines (in fact all ratios) the same. Thus, sine of each of the angles 30°, 150°, 390°, 510°, -330° , -210° , etc. is equal to $\frac{1}{2}$. Similarly, if $\cos\theta$ be given, equal to $\frac{1}{\sqrt{2}}$ say, θ may have any of the values $+45^\circ$, $+315^\circ$, $+405^\circ$, -315° , -45° , etc.; or else, if $\tan\theta = \sqrt{3}$, θ may have any of the values 60° , 240° , 420° , -300° , etc.

It is very convenient for the solution of trigonometrical equations, as also for other purposes, to obtain a general expression in a compact form embracing all angles, the trigonometrical ratios of which have a given value.

58. General expression of all angles, one of whose trigonometrical ratios is zero.

, If the sine of an angle be zero, from definition, the length of the perpendicular from any point of one of its arms upon another is zero, so that the two arms must be in the same straight line. Evidently, therefore, such angles must be zero, or some multiple of π , odd or even.

Thus, if $\sin \theta = 0$, then $\theta = n\pi$, n being zero, or any integer, positive or negative.

When the cosine of an angle is zero, the projection of any length along one arm upon another is zero, and so the two arms must be at right angles to one another. The angles must therefore be evidently either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ or differ from these by complete revolutions; in other words, the angle may be any odd multiple of $\frac{\pi}{2}$.

Thus, if $\cos \theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$.

n being zero, or any integer, positive or negative.

Again, if $\tan \theta = 0$, then its numerator $\sin \theta$ is also zero; and so $\theta = n\pi$.

Similarly, if cot $\theta = 0$, then $\cos \theta = 0$;

and so
$$\theta = (2n+1) \frac{\pi}{2}$$
.

Note. The ratios cosec θ or sec θ can never be zero, for they can never be numerically less than unity.

59. General expression of angles having the same sine (or cosecant).

Let a be any angle positive or negative such that its sine is equal to a given quantity k (numerically not greater than 1); for fixing up the idea, and for the sake of convenience in practice, the smallest positive angle having its sine for the given quantity k is taken as a. Let θ be any other angle whose sine is equal to k.

Then, $\sin \theta = \sin \alpha$, or, $\sin \theta - \sin \alpha = 0$,

or, $2 \sin \frac{1}{2} (\theta - a) \cos \frac{1}{2} (\theta + a) = 0$.

 $\therefore \quad \text{either sin } \frac{1}{2} \left(\theta - a \right) = 0,$

i.e., $\frac{1}{2}(\theta - a) = \text{any multiple of } n = mn$, ... (1)

or, else $\cos \frac{1}{2} (\theta + \alpha) = 0$,

i.e.,
$$\frac{1}{2}(\theta + a) = \text{any odd multiple of } \frac{\pi}{2} = (2m+1) \cdot \frac{\pi}{2} \cdot \cdots$$
 (2)

From (1),
$$\theta - a = 2m\pi$$
, i.e., $\theta = a + 2m\pi$ (3)

From (2),
$$\theta + a = (2m+1) \pi$$
, i.e., $\theta = -a + (2m+1) \pi$... (4)

Combining (3) and (4), $\theta = (-1)^n a + n\pi$... (5) where n is zero, or any integer, positive or negative, odd or even.

If cosec $\theta = \operatorname{cosec} a$, then $\sin \theta = \sin a$; hence all angles having the same cosecant as that of a are also given by the expression (5).

Thus, all angles having the same sine or cosecant as that of a are given by $2n\pi + a$ and $(2n + 1)\pi - a$,

or,
$$n\pi + (-1)^n\alpha$$
.

60. General expression of angles having the same cosine (or secant).

Let a be the smallest positive angle such that its cosine is equal to a given quantity k (numerically ≥ 1); and let θ be any other angle whose cosine is equal to k.

Then, $\cos \theta = \cos a$.

or,
$$\cos a - \cos \theta = 0$$
,

$$\therefore$$
 2 sin $\frac{1}{2}(\theta + a) \sin \frac{1}{2}(\theta - a) = 0.$

$$\therefore \text{ either sin } \frac{1}{2} (\theta + a) = 0,$$

i.e.,
$$\frac{1}{2}(\theta + a) = \text{any multiple of } \pi = n\pi$$
 ... (1)

or else,
$$\sin \frac{1}{2}(\theta - a) = 0$$
,

i.e.,
$$\frac{1}{2}(\theta - a) = \text{any multiple of } \pi = n\pi.$$
 (2)

From (1),
$$\theta + a = 2n\pi$$
, or, $\theta = 2n\pi - a$, ... (3)

From (2),
$$\theta - a = 2n\pi$$
, or, $\theta = 2n\pi + a$ (4)

From (3) and (4), we have $\theta = 2n\pi \pm a$, ... (5) where n is zero, or any integer, positive or negative.

It is also evident as in the previous case that all angles having the same secant as that of a are also included in the expression (5).

Hence, all angles having the same cosine or secant as that of a are given by

$$2n\pi \pm \alpha$$

n being zero, or any integer, positive or negative.

Note. As in Art. 59, instead of taking the smallest positive angle, we might take a to be any one angle having for its cosine the given quantity k. The general values of θ satisfying $\cos \theta = \cos a$ as obtained above, would not be affected at all.

61. General expression of all angles having the same tangent (or cotangent).

Let a be the smallest positive angle such that its tangent is equal to a given quantity k; and let θ be any other angle whose tangent is equal to k.

Then, $\tan \theta = \tan a$,

or,
$$\frac{\sin \theta}{\cos \theta} - \frac{\sin \alpha}{\cos \alpha} = 0$$
,

or,
$$\frac{\sin \theta \cos \alpha - \cos \theta \sin \alpha}{\cos \theta \cos \alpha} = 0,$$

or,
$$\frac{\sin (\theta - a)}{\cos \theta \cos a} = 0$$
.

$$\therefore \sin (\theta - a) = 0,$$

i.e., $\theta - a = \text{any multiple of } \pi = n\pi$.

$$\therefore \quad \theta = a + n\pi. \qquad \cdots \quad (1)$$

The factor $\frac{1}{\cos \theta} \frac{1}{\cos a}$ cannot be zero, for cosine of an angle cannot have an infinitely large value.

It is also evident as in the previous case that all angles having the same cotangent as that of α are given by the expression (1).

Hence, all angles having the same tangent or cotangent as that of a are given by

$$n\pi + \alpha$$

n being zero, or any integer, positive or negative.

Note. The remark below Art. 60 is applicable here also.

62. Special cases.

From Art. 59, considering both cases when n is odd or even, it may be easily seen that

if
$$\sin \theta = 1 = \sin \frac{\pi}{2}$$
, $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$;
and if $\sin \theta = -1 = \sin \left(-\frac{\pi}{2}\right)$, $\theta = 2n\pi - \frac{\pi}{2} = (4n-1)\frac{\pi}{2}$
or, $= (4k+3)\frac{\pi}{2}$,

where n (or k=n-1) is zero, or any integer, positive or negative.

Similarly, from Art. 60, it may be seen that

if
$$\cos \theta = 1$$
, $\theta = 2n\pi$

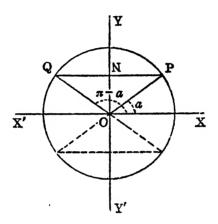
and if
$$\cos \theta = -1$$
, $\theta = (2n+1)\pi$,

n being zero, or any integer, positive or negative.

These are the usual forms in which the above special cases are used in practice.

63. Geometrical Treatment.

(i) Geometrical construction of an angle whose sine (or cosecant) is given, and to obtain a general expression of all such angles.



Let the sine of an angle be given equal to 'a'.

Taking the perpendicular lines XOX' and YOY' for reference, draw a circle of unit radius with centre O.

Measure off ON = a along OY (or along OY' if a be negative). Through N draw a straight line PNQ parallel to XOX' meeting the circle at P and Q.

Then, $\angle POX = a$ say, is one of the required angles, for $\sin a = \sin OPN = \frac{ON}{OP} - a$.

Another angle with the same sine, as is apparent from the figure, is $\angle QOX = \pi - a$ (or $3\pi - a$ if a = ON be negative, which is trigonometrically the same as n - a).

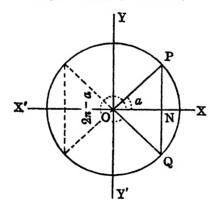
'a' being given in magnitude and sign, the position of N on YOY' is fixed and thus in one revolution, i.e. from 0 to

 2π there are, as is clear from the figure, only two angles a and $\pi - a$ having the given sine.*

Now, the addition or subtraction of any multiple of 2π makes no difference in the values of the trigonometrical ratios of an angle (See Art. 28).

Hence, all the angles having the same sine as that of a are contained in the formulæ $2m\pi + a$ and $2m\pi + \pi - a$ i.e., $(2m+1)\pi - a$, where m is zero, or any integer, positive or negative. Both the sets of angles are evidently included in the formula $n\pi + (-1)^n a$, n being zero, or any integer, positive or negative.

(ii) Angles with given cosine (or secant).



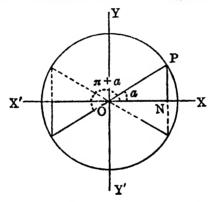
Let the given cosine be 'a'. As before, measure off ON=a along OX (or along OX' if 'a' be negative), and through N draw PNQ parallel to YOY' to meet the circle with centre O and radius unity, at P and Q.

^{*} In the same quadrant there cannot be two distinct angles (without being cotorminals) having the same sine, for the corresponding triangles will then be congruent.

Let $\angle POX = a$. Then, a is a required angle. Also from the figure, the only angles in the first four quadrants which have the given cosine are a and $2\pi - a$.

Adding or subtracting multiples of 2π to these, all the angles having the same cosine as that of a are given by $2m\pi + a$ or $2m\pi + 2\pi - a$, both of which are included in the formula $2n\pi \pm a$, n being zero, or any integer, positive or negative.

(iii) Angles with given tangent (or cotangent).



Let 'a' be the given tangent. Along OX or OX' measure off ON of unit length, and then measure off NP perpendicular to it of length whose numerical value is 'a'. If 'a' be positive, both ON and NP will be positive, or both will be negative, and so the $\angle XOP$ will be either in the first or in the third quadrant. If 'a' be negative, the angle will be either in the second or in the fourth quadrant. In any case there are only two angles, within one revolution, i.e., from 0 to 2π as is apparent from the figure, with the given tangent.*

^{*}The ratio PN:ON being given, and the included angle PNO being right, the triangle PNO constructed remains always similar to itself and so in the same quadrant $\angle PON$ of the triangle is unique.

One of the angles being a, the other is evidently (from the figure) n+a. Adding or subtracting multiples of 2π , all the angles having the same tangent as that of a are given by 2mn+a or 2mn+n+a both of which are included in the formula nn+a where n is zero, or any integer, positive or negative, odd or even.

Ex. 1. Solve
$$2(\cos^2\theta - \sin^2\theta) = 1$$
.

The given equation can be written as

$$2\cos 2\theta = 1$$
. $\cos 2\theta = \frac{1}{2} = \cos \frac{1}{3}\pi$.

$$\therefore 2\theta = 2n\pi \pm \frac{1}{2}\pi. \qquad \therefore \theta = n\pi \pm \frac{1}{2}\pi.$$

Note. It may be observed that a trigonometrical equation can be solved in several ways; and the results though different in forms will give the same series of angles. To illustrate this we work out the above example in another way.

The equation can also be written in the form

$$2(\cos^2\theta - 1 + \cos^2\theta) = 1$$
, or, $4\cos^2\theta = 3$.

$$\therefore \cos \theta = \pm \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6}, \text{ or, } \cos \frac{5\pi}{6}.$$

$$\theta = 2m\pi \pm \frac{\pi}{6}$$
; or, $2m\pi \pm \frac{5\pi}{6}$.

Now,
$$2m\pi \pm \frac{5\pi}{6} = (2m+1)\pi - \frac{\pi}{6}$$
 or, $(2m-1)\pi + \frac{\pi}{6}$

All the four sets of solutions, m being any integer, can be included in the expression $n\pi \pm \frac{1}{6}\pi$, in which form the result has already been obtained by the previous process.

Ex. 2. Solve $4 \cos^2 x + 6 \sin^2 x = 5$.

The equation can be written as

$$4\cos^2 x + 6\sin^2 x = 5(\sin^2 x + \cos^2 x).$$

$$\therefore \sin^2 x = \cos^2 x, \text{ or, } \tan^3 x = 1.$$

$$\therefore \tan x = \pm 1. \quad \therefore \quad x = n\pi \pm \frac{1}{4}\pi.$$

Note. Equations of the form a cos*x+b sin*x=c can be easily solved by the above method, or by expressing sine in terms of cosine or cosine in terms of sine.

Ex. 3. Solve
$$2 \sin^2 x + \sin^2 2x = 2$$
. [C. U. 1940]

The given equation can be written as

$$2(1-\sin^2 x)-\sin^2 2x=0$$
, or, $2\cos^2 x-4\sin^2 x\cos^2 x=0$.

or,
$$2\cos^2 x (1-2\sin^2 x) = 0$$
, or, $\cos^2 x \cos 2x = 0$.

 \therefore either cos x=0, i.e., $x=n\pi+\frac{1}{2}\pi$,

or,
$$\cos 2x = 0$$
, i.e., $2x = 2n\pi \pm \frac{1}{2}\pi$. $\therefore \quad x = n\pi \pm \frac{1}{4}\pi$.

Ex. 4. Solve
$$\cos \theta - \sin \theta = \frac{1}{\sqrt{2}}$$
.

Dividing both sides of the equation by $\sqrt{1^2 + 1^2}$, i.e., $\sqrt{2}$, we have

$$\cos \theta \cdot \frac{1}{\sqrt{2}} - \sin \theta \cdot \frac{1}{\sqrt{2}} = \frac{1}{2},$$

i.e., $\cos \theta \cos \frac{1}{2}\pi - \sin \theta \sin \frac{1}{2}\pi = \frac{1}{2}$.

$$\therefore \cos (\theta + \frac{1}{4}\pi) = \cos \frac{1}{3}\pi. \quad \therefore \quad \theta + \frac{1}{4}\pi = 2n\pi \pm \frac{1}{3}\pi.$$

$$\theta = 2n\pi + \frac{1}{12}\pi$$
, or, $2n\pi - \frac{7}{12}\pi$.

Note. Extraneous solutions.

In general, as pointed out in Ex. 1 above, the same trigonometrical equation may be solved by different methods, and the forms of the result we arrive at, though apparently different in some cases, are ultimately equivalent. In some cases, however, we may be tempted to solve a trigonometrical equation by methods which have flaws in them leading to solutions which include in addition to the correct solutions, some extraneous solutions which do not satisfy the given equation. The given equation which is of the type $a\cos\theta+b\sin\theta=c$ is an example. We proceed to demonstrate it as follows:

Here,
$$\cos \theta - \frac{1}{\sqrt{2}} = \sin \theta$$
.

 $\therefore \cos^2\theta - \sqrt{2}\cos\theta + \frac{1}{2} = \sin^2\theta = 1 - \cos^2\theta,$ where $2\cos^2\theta - \sqrt{2}\cos\theta - \frac{1}{2} = 0.$

$$\therefore \cos \theta = \frac{\sqrt{2} \pm \sqrt{2} + 4}{4} = \frac{1 \pm \sqrt{3}}{2 \sqrt{2}} = \cos \frac{\pi}{12}, \text{ or, } \cos \frac{7\pi}{12}.$$

 $\theta = 2n\pi \pm \frac{1}{12}\pi, \text{ or, } 2n\pi \pm \frac{7}{12}\pi.$

But it can be easily seen on substitution that

 $2n\pi - \frac{1}{1}\pi$ and $2n\pi + \frac{7}{1}\pi$ do not satisfy the given equation. The crror in the method lies in squaring the equation as we have done; for the squared equation includes the equation $\cos \theta - \frac{1}{\sqrt{2}} = -\sin \theta$, i.e., $\cos \theta + \sin \theta = \frac{1}{\sqrt{2}}$ of which the solutions are $2n\pi - \frac{1}{1}\pi\pi$ and $2n\pi + \frac{7}{1}\pi\pi$.

Equations of this type are therefore best solved as in the next example, and not by squaring.

Thus, while solving any trigonometrical equation it is always advisable to verify the roots obtained; for thereby extraneous roots, if any, can be easily detected.

Ex. 5. Solve
$$a \cos \theta + b \sin \theta = c$$
. $(c > \sqrt{a^2 + b^2})$

Put $a = r \cos a$, $b = r \sin a$, choosing the smallest positive value of a, keeping r positive.

Then,
$$r = \sqrt{a^2 + b^2}$$
 and $\sin a = \frac{b}{\sqrt{a^2 + b^2}}$
and $\cos a = \frac{a}{\sqrt{a^2 + b^2}}$

The signs of a and b will determine the quadrant in which a lies, and a and b being given, r and a are definitely known.

The equation now becomes

$$r \cos (\theta - a) = c$$
,
or, $\cos (\theta - a) = \frac{c}{\sqrt{a^2 + h^2}} = \cos \beta$,

where β is the smallest positive angle whose cosine is $\sqrt{a^2+b^2}$, and a, b, c being known, β is also known.

Hence,
$$\theta - a = 2n\pi \pm \beta$$
, or, $\theta = 2n\pi + \alpha \pm \beta$.

Note. An angle which is introduced in a trigonometrical work to facilitate calculations is called a subsidiary angle. Thus, a and β are here subsidiary angles.

Ex. 6. Solve $4 \cos x + 5 \sin x = 5$, given $\tan 51^{\circ} 21' = \frac{5}{4}$. Dividing both sides of the given equation by $\sqrt{4^2 + 5^2}$, i.e., by $\sqrt{41}$, we get

$$\frac{4}{\sqrt{41}}\cos x + \frac{5}{\sqrt{41}}\sin x = \frac{5}{\sqrt{41}}.$$
 (1)

Since, $\tan 51^{\circ} 21' = \frac{5}{4}$,

$$\therefore \sin 51^{\circ} 21' = \frac{5}{\sqrt{41}}, \cos 51^{\circ} 21' = \frac{4}{\sqrt{41}}.$$

... (1) reduces to $\cos x \cos 51^{\circ} 21' + \sin x \sin 51^{\circ} 21' = \sin 51^{\circ} 21'$

or,
$$\cos(x-51^{\circ}21') = \sin 51^{\circ}21' = \cos 38^{\circ}39'$$
.

$$\therefore x - 51^{\circ} 21' = 2n\pi \pm 38^{\circ} 39'.$$

$$\therefore x = 2n\pi + 90^{\circ}$$
, or, $2n\pi + 12^{\circ} 42'$.

Ex. 7. (i) Solve $2 \sin^2 x + \sin^2 2x = 2$ for $-\pi < x < \pi$.

From Ex. 3 above, we see that
$$x = n\pi + \frac{1}{2}\pi$$
 ... (1)

or,
$$x = n\pi \pm \frac{1}{4}\pi$$
. ... (2)

Putting n=0, -1 in (1), we get $x=\frac{1}{2}\pi$, $-\frac{1}{2}\pi$, which lie in the given interval. Putting n=0, 1, -1 in (2), we get $x=\pm\frac{1}{4}\pi$, $\frac{3}{4}\pi$, $-\frac{3}{4}\pi$ which also lie in the given interval.

Hence, the required values of x are $\pm \frac{1}{4}\pi$, $\pm \frac{1}{2}\pi$, $\pm \frac{3}{4}\pi$.

(ii) Solve $\cos \theta + \sqrt{3} \sin \theta = 2$

for
$$-2\pi < \theta < 2\pi$$
 and $3\pi < \theta < 5\pi$.

Dividing both sides of the equation by $\sqrt{1+3}$, i.e., 2, we have

$$\cos \theta \cdot \frac{1}{2} + \sin \theta \cdot \frac{\sqrt{3}}{2} = 1,$$

i.e., $\cos \theta \cdot \cos \frac{1}{3}n + \sin \theta \cdot \sin \frac{1}{3}n = 1$,

i.e., $\cos (\theta - \frac{1}{2}\pi) = 1$.

$$\therefore \quad \theta - \frac{1}{3}\pi = 2n\pi, \ i.e., \ \theta = 2n\pi + \frac{1}{3}\pi.$$

Putting n=0, -1, we get $\theta = \frac{1}{3}\pi$, $-\frac{2}{3}\pi$ which lie in the 1st interval.

Again, putting n=1, 2, we get $\theta = \frac{7}{8}\pi$, $\frac{1.3}{8}\pi$, which lie in the 2nd interval.

Ex. 8. Solve $\tan ax = \cot bx$.

Here, $\tan ax = \cot bx = \tan (\frac{1}{2}\pi - bx)$.

$$\therefore ax = n\pi + \frac{1}{2}\pi - bx.$$

$$\therefore \quad x = \frac{2n+1}{a+b} \cdot \frac{\pi}{2} \cdot$$

Examples XI

Solve the following equations (Ex. 1 to 23):—

- 1. $\cot^2 x + \csc^2 x = 3$.
- 2. (i) $2 \cos^2 \theta + 4 \sin^2 \theta = 3$.

(ii)
$$\tan^2 \theta = 3 \csc^2 \theta - 1$$
.

[C. U. 1939]

- 3. $\tan x \cot x = \csc x$.
- 4. $\cot x \cot 2x = 2$.
- $2 \sin \theta \tan \theta + 1 = \tan \theta + 2 \sin \theta$.
- $\sin 5\theta + \sin \theta = \sin 3\theta$. R
- 7. $\sin m\theta + \sin n\theta = 0$.
- 8. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$.
- 9. $\cot 2x = \cos x + \sin x$.
- 10. $\sin x + \cos x = \sqrt{2}$, for $-\pi < x < \pi$.
- 11. $\sin 2x \tan x + 1 = \sin 2x + \tan x$.
- 12. $\cot x \tan x = 2$.

[C. U. 1934, '37]

- 13. $\sin x + \sqrt{3} \cos x = \sqrt{2}$.
- [C. U. 1938, '47]

- 14. $2 \sin x \sin 3x = 1$.
- 15. $\sin \theta + 2 \cos \theta = 1$.

[C. U. 1933]

- 16. $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$.
- 17. $\tan (\frac{1}{4}\pi + \theta) + \tan (\frac{1}{4}\pi \theta) = 4$. [C. U. 1949]

- 18. $\tan x + \tan 2x + \tan x \tan 2x = 1$. [C. U. 1941, '45]
- 19. $\cos \theta + \sqrt{3} \sin \theta = \sqrt{2}$.

[C. U. 1944]

- 20. $\sqrt{3} \cos x + \sin x = 1$, for $-2\pi < x < 2\pi$.
- 21. $\cos 2x = \cos x \sin x$.
- 22. $2 \cot x + \sin x = 2 \csc x$.
- 23. $\cos x + \sin x = \cos 2x + \sin 2x$. [C. U. 1943]

- 24. Solve $2 \sin^2 x + \sin x = 3$; and find all the angles between 0° and 1000° which satisfy it.
- Find the solution of the equations (general solution is not required)

$$\tan x + \tan y = 2$$

$$2 \cos x \cos y = 1$$

- 26. If $\tan ax \tan bx = 0$, show that the values of x form a series in A.P.
 - 27. Solve
 - (i) $\cos 3x + \cos 2x + \cos x = 0$. [C. U. 1941, '49]
 - (ii) $\cos 9x \cos 7x = \cos 5x \cos 3x$, $-\frac{1}{4}\pi \le x \le \frac{1}{4}\pi$.
 - (iii) $\tan x + \tan 2x + \tan 3x = 0$. [A. I. 1941]
 - (iv) $\cos x \sin x = \cos a + \sin a$. [B. II. U. 1938]
 - (v) $\cos^3 x \cos x \sin x \sin^3 x = 1$.
 - (vi) $\cos 6x + \cos 4x = \sin 3x + \sin x$.
 - (vii) $\frac{\sin \alpha}{\sin 2x} + \frac{\cos \alpha}{\cos 2x} = 2$.
 - 28. Salve 5 cos $\theta + 2 \sin \theta = 2$, given tan 68° $12' = 2\frac{1}{2}$.
- 29. Find those pairs of solutions of the following equations which correspond to positive solutions less than 2π of each individual equation:—
 - (i) $\sin (\alpha \beta) = 0$; $\sin (\alpha + \beta) = 1$.
 - (ii) $\sin (a \beta) = \cos (a + \beta) = \frac{1}{2}$.
- 30. If $\sin A = \sin B$, $\cos A = \cos B$, prove that either A and B are equal or they differ by some multiple of four right angles. [C. U. 1935]
- 31. Show that the three equations $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$, $\tan^2\theta = \tan^2\alpha$ are all identical and the solution is always $n\pi \pm a$.
- 32. Show that the same two series of angles are given by the equations

$$x + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6}$$
 and $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}$.

CHAPTER XII

INVERSE CIRCULAR FUNCTIONS

64. The equation $\sin \theta = x$ means that θ is an angle whose sine is x. It is often convenient to express this statement inversely by writing $\theta = \sin^{-1}x$. Thus, the symbol $\sin^{-1}x$ denotes an angle whose sine is x. Hence, $\sin^{-1}x$ is an angle, whereas $\sin \theta$ is a number. The two relations $\sin \theta = x$ and $\theta = \sin^{-1}x$ are identical; if one is given the other follows. The symbol $\sin^{-1}x$ is usually read as "sine inverse x". Sometimes it is also denoted by $arc \sin x$.

Note. $\sin^{-1}x$ must not be confused with $(\sin x)^{-1}$, i.e., $\frac{1}{\sin x}$.

65. We know that if θ be any one angle whose sine is equal to x, then sines of all the angles given by $n\pi + (-1)^n\theta$ are equal to x. Hence, $\sin^{-1}x$ has got an infinite number of values, and as such, $\sin^{-1}x$ is a multiple-valued function.

Hence, the general value of $\sin^{-1}x = n\pi + (-1)^n \sin^{-1}x$ where on the rigt-hand side $\sin^{-1}x$ stands for any particular angle whose sine is x.

Similarly, the general value of

$$\cos^{-1}x = 2n\pi \pm \cos^{-1}x$$

and of
$$tan^{-1}x = n\pi + tan^{-1}x$$
.

The smallest numerical value, either positive or negative, of θ is called the *principal value* of $\sin^{-1}x$. Thus, the principal value of $\sin^{-1}\frac{1}{2}$ is 30°. If corresponding to the same ratio, there are two numerically equal angles, one positive and the other negative, it is customary to take the positive angle as the principal value; thus, the principal value of $\cos^{-1}\frac{1}{2}$ is 60°, and not (.60°) although $\cos(-60^\circ)=\frac{1}{3}$.

In all numerical examples, the principal value is generally taken.

 $\cos^{-1}x$, $\tan^{-1}x$, $\csc^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ have similar significance and all properties as those of $\sin^{-1}x$. These expressions are called Inverse Circular Functions.

66. If
$$\sin \theta = x$$
, then $\theta = \sin^{-1} x$, i.e., $\theta = \sin^{-1} \sin \theta$.

Similarly, $\theta = \cos^{-1} \cos \theta = \tan^{-1} \tan \theta$; etc.

Again, if $\theta = \sin^{-1}x$, $\sin \theta = x$, i.e., $\sin \sin^{-1}x = x$.

Similarly, $\cos \cos^{-1} x = x$; $\tan \tan^{-1} x = x$; etc.

Also, we have

$$\csc^{-1}x = \sin^{-1}\frac{1}{x}$$
; $\cot^{-1}x = \tan^{-1}\frac{1}{x}$; $\sec^{-1}x = \cos^{-1}\frac{1}{x}$.

Let $\csc^{-1}x = \theta$; then $\csc \theta = x$.

$$\therefore \sin \theta = \frac{1}{\cos \cot \theta} = \frac{1}{x}.$$

Hence, $\theta = \sin^{-1} \frac{1}{x}$, and therefore, $\csc^{-1} x = \sin^{-1} \frac{1}{x}$.

In the same way we have, $\csc^{-1}\frac{1}{x} = \sin^{-1}x$.

The other relations follow similarly.

67. As all the trigonometrical ratios can be expressed in terms of any one, similarly all the inverse trigonometrical ratios can be expressed in terms of any one inverse ratio.

Thus, let $\sin^{-1}x = \theta$; then $\sin \theta = x$,

$$\therefore \cos \theta = \sqrt{1-x^2} ; \tan \theta = \frac{x}{\sqrt{1-x^2}} ; \cot \theta = \frac{\sqrt{1-x^2}}{x} ;$$

$$\sec \theta = \frac{1}{\sqrt{1-x^2}} \text{ and } \csc \theta = \frac{1}{x} .$$

$$\theta = \sin^{-1} x = \cos^{-1} \sqrt{1 - x^2} = \tan^{-1} \frac{x}{\sqrt{1 - x^2}}$$

$$= \cot^{-1} \frac{\sqrt{1 - x^2}}{x} = \sec^{-1} \frac{1}{\sqrt{1 - x^2}} = \csc^{-1} \frac{1}{x}.$$

68. To prove that

(i)
$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$
.

(ii)
$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$
.

(iii)
$$\csc^{-1}x + \sec^{-1}x = \frac{\pi}{2}$$
.

(i) Let $\sin^{-1}x = \theta$; then $\sin \theta = x$.

Now, $\sin \theta = \cos \left(\frac{1}{2}\pi - \theta\right)$.

$$\therefore \cos \left(\frac{1}{2}\pi - \theta\right) = x \text{ and hence } \cos^{-1} x = \frac{1}{2}\pi - \theta.$$

Therefore, $\sin^{-1} x + \cos^{-1} x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$.

(ii) Let $\tan^{-1}x = \theta$; then $\tan \theta = x$.

Now, $\tan \theta = \cot (\frac{1}{2}\pi - \theta)$.

$$\therefore \cot \left(\frac{1}{2}\pi - \theta\right) = x. \quad \therefore \cot^{-1}x = \frac{1}{2}\pi - \theta.$$

$$\therefore \tan^{-1}x + \cot^{-1}x = \theta + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi.$$

(iii) Let $\csc^{-1}x = \theta$; then $\csc \theta = x$.

Now, cosec $\theta = \sec(\frac{1}{3}\pi - \theta)$.

$$\therefore$$
 sec $(\frac{1}{2}\pi - \theta) = x$. \therefore sec⁻¹ $x = \frac{1}{2}\pi - \theta$.

$$\therefore$$
 $\csc^{-1}x + \sec^{-1}x = 0 + \frac{1}{2}\pi - \theta = \frac{1}{2}\pi$.

69. To prove that

(i)
$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

(ii)
$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$

Let $\tan^{-1}x = a$; and $\tan^{-1}y = \beta$;

then $\tan a = x$; and $\tan \beta = y$.

Now,
$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{x + y}{1 - xy}$$

$$\therefore a+\beta=\tan^{-1}\frac{x+y}{1-xy},$$

i.e.,
$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

Again,
$$\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha} = \frac{x - y}{1 + xy}$$

$$\therefore \quad a-\beta=\tan^{-1}\frac{x-y}{1+xy},$$

i.e.,
$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$
.

Note. It can be easily proved as above that

$$\cot^{-1}x \pm \cot^{-1}y = \cot^{-1}\frac{xy \mp 1}{y \pm x}.$$

70. To prove that

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{x+y+z-xyz}{1-yz-zx-xy}$$

Let $\tan^{-1} x = a$; $\tan^{-1} y = \beta$; $\tan^{-1} z = \gamma$.

$$\therefore$$
 tan $\alpha = x$, tan $\beta = y$, tan $\gamma = z$.

Now, $\tan (\alpha + \beta + \gamma)$

$$= \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \beta \tan \gamma - \tan \gamma \tan \alpha - \tan \alpha \tan \beta}$$

$$= \frac{x + y + z - xyz}{1 - yz - zx - xy}$$

Hence,
$$\alpha + \beta + \gamma = \tan^{-1} \frac{x + y + z - xyz}{1 - yz - zx - xy}$$

Since, $a + \beta + \gamma = \tan^{-1}x + \tan^{-1}y + \tan^{-1}z$, the required result follows.

Note. This relation can also be deduced by applying twice the formula of Art. 69. Thus,

Left side =
$$(\tan^{-1}x + \tan^{-1}y) + \tan^{-1}s$$

= $\tan^{-1}\frac{x+y}{1-x^{st}} + \tan^{-1}s$; now again apply Art. 69.

71. In fact, for most of the formulæ involving ordinary circular functions, corresponding relations connecting the inverse circular functions can be easily deduced. In addition to those given above, some are illustrated in the following examples:

Ex. 1. Show that

(i)
$$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}.$$

(ii)
$$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy + \sqrt{(1-x^2)(1-y^2)})$$
.

(i) Let $\sin^{-1}x = a$. \therefore $\sin a = x$ and $\cos a = \sqrt{1-x^2}$; also let $\sin^{-1}y = \beta$. \therefore $\sin \beta = y$ and $\cos \beta = \sqrt{1-u^2}$.

Now,
$$\sin (a \pm \beta) = \sin a \cos \beta \pm \cos a \sin \beta$$

= $x \sqrt{1 - y^2} \pm y \sqrt{1 - x^2}$.
 $\therefore a + \beta = \sin^{-1} \{x \sqrt{1 - y^2} + y \sqrt{1 - x^2}\}$.

$$\dots \quad \mathbf{d} \pm \mathbf{p} - \mathbf{s} \mathbf{m} \quad \{x \, \mathbf{v} \, \mathbf{1} - y^- \pm y \, \mathbf{v} \, \mathbf{1} - x^-\}.$$

Since, $a \pm \beta = \sin^{-1} x \pm \sin^{-1} y$, the required result follows.

(ii) These relations follow similarly from the value of $\cos (a \pm \beta)$.

Ex. 2. Show that

(i)
$$2 \sin^{-1} x = \sin^{-1} (2x \sqrt{1-x^2})$$
.

(ii)
$$2 \cos^{-1} x = \cos^{-1} (2x^2 - 1)$$
.

(iii)
$$2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$$
.

(i) Let $\sin^{-1}x = a$. $\therefore \sin a = x$, $\cos a = \sqrt{1-x^2}$.

Now, $\sin 2a = 2 \sin a \cos a = 2x \sqrt{1-x^2}$.

$$\therefore$$
 $2a = \sin^{-1}(2x \sqrt{1-x^2}).$

Since, $a = \sin^{-1}x$, the required result follows.

(ii) & (iii). These relations follow similarly from the corresponding values of $\cos 2a$ in terms of $\cos a$ and of $\tan 2a$ in terms of $\tan a$. [See Art. 43]

Note. The above three relations can also be deduced by putting x for y in the values of $\sin^{-1}x + \sin^{-1}y$, $\cos^{-1}x + \cos^{-1}y$ and $\tan^{-1}x + \tan^{-y}y$.

Ex. 3. Show that

(i)
$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$$
.

(ii)
$$3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x)$$
.

(iii)
$$3 \tan^{-1} x = \tan^{-1} \frac{3x - x^3}{1 - 3x^2}$$
 [C. U. 1938]

(i) Let $\sin^{-1}x = \theta$; then $\sin \theta = x$.

Now,
$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta = 3x - 4x^3$$
.

30, i.e.,
$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$$
.

(ii) & (iii). These relations follow similarly from the corresponding values of $\cos 3\theta$ in terms of $\cos \theta$ and of $\tan 3\theta$ in terms of $\tan \theta$. [See Art. 44]

Note. The result (iii) may also be deduced by putting y=z=x in the formula of Art. 70.

Ex. 4. Show that

2
$$\tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2} = \tan^{-1} \frac{2x}{1-x^2}$$

Let $\tan^{-1}x = \theta$, \therefore $\tan \theta = x$.

Since,
$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2x}{1 + x^2}$$
, [Art. 45, Ex. 1]

$$\therefore 2\theta, i.e., 2 \tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}.$$

Since,
$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - x^2}{1 + x^2}$$

and
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2x}{1 - x^2}$$

the remaining relations follow similarly.

Ex. 5. Show that

$$tan^{-1}\frac{a-b}{1+ab}+tan^{-1}\frac{b-c}{1+bc}+tan^{-1}\frac{c-a}{1+ca}=0.$$

1st term of left side = $tan^{-1}a - tan^{-1}b$ [By Art. 69 (ii)],

2nd
$$\cdots = \tan^{-1}b - \tan^{-1}c$$
,

$$3rd \cdots \cdots = tan^{-1}c - tan^{-1}a.$$

Hence, adding up the three terms, the required result follows.

Ex. 6. Show that

$$2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{32}{43}$$

Since,
$$2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$$
,

[See Ex. 4]

$$\therefore 2 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{5}} = \tan^{-1} \frac{5}{15}.$$

$$\therefore \text{ left side} = \tan^{-1} \frac{5}{19} + \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{\frac{5}{19} + \frac{1}{4}}{1 - \frac{5}{19}} = \tan^{-1} \frac{39}{49}.$$

Ex. 7. Solve

$$\sin^{-1}\frac{2a}{1+a^2} + \sin^{-1}\frac{2b}{1+b^2} = 2 \tan^{-1}x.$$

[C. U. 1947]

Since,
$$\sin^{-1} \frac{2x}{1+x^2} = 2 \tan^{-1} x$$
, [See Ex. 4]

$$\therefore \quad \text{left side} = 2 \tan^{-1} a + 2 \tan^{-1} b.$$

... the equation reduces to $2 \tan^{-1} x = 2 \tan^{-1} a + 2 \tan^{-1} b.$

$$\tan^{-1}x = \tan^{-1}a + \tan^{-1}b = \tan^{-1}\frac{a+b}{1-ab}.$$

$$x = \frac{a+b}{1-ab}.$$

Ex. 8. Solve

$$\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$$
.

Left side =
$$\tan^{-1} \frac{x-1}{x-2} + \frac{x+1}{x+2} = \tan^{-1} \frac{2x^2-4}{-3}$$
.

... the equation reduces to

$$\tan^{-1}\frac{2x^2-4}{-3}=\frac{\pi}{4}=\tan^{-1}1.$$

$$\frac{2x^2-4}{-3}=1 \text{ or, } 2x^2=1 \text{ or, } x=\pm \frac{1}{\sqrt{2}}.$$

Examples XII

Prove that (Ex. 1 to 17):-

1. (i)
$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{1}{4}\pi$$
.

(ii)
$$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^3}{1-3x^2}$$
.

(iii)
$$\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18} = \cot^{-1} 3$$
.

2.
$$\tan^{-1} \frac{9}{11} + \cot^{-1} \frac{24}{7} = \tan^{-1} \frac{1}{2}$$
.

3.
$$\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$$

= $2(\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2})$.

4. (i)
$$\tan^{-1}x + \cot^{-1}(x+1) = \tan^{-1}(x^2 + x + 1)$$
.

(ii)
$$\tan^{-1} \frac{1}{p+q} + \tan^{-1} \frac{q}{p^2 + pq + 1} = \tan^{-1} \frac{1}{p}$$
.
 $-1 \frac{a-b}{1+ab} + \tan^{-1} \frac{b-c}{1+bc}$.

6.
$$\tan^{-1} \frac{3}{5} + \sin^{-1} \frac{9}{5} = \tan^{-1} \frac{27}{11}$$
.

7.
$$\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{5} = \frac{1}{4}\pi$$
.

8.
$$2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2} = \frac{1}{4}\pi$$
.

9. (i)
$$\sin (2 \sin^{-1} x) = 2x \sqrt{1-x^2}$$
.

(ii)
$$\{\cos(\sin^{-1}x)\}^2 = \{\sin(\cos^{-1}x)\}^2$$
.

10.
$$\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$$

11.
$$\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \frac{1-x}{1+x}$$
 [C. U. 1943]

12.
$$\sin^{-1} \sqrt{\frac{x-b}{a-b}} = \cos^{-1} \sqrt{\frac{a-x}{a-b}} = \tan^{-1} \sqrt{\frac{x-b}{a-x}}$$

13.
$$\tan^{-1} \frac{a-b}{1+ab} + \tan^{-1} \frac{b-c}{1+bc} + \tan^{-1} \frac{c-a}{1+ca}$$

= $\tan^{-1} \frac{a^2-b^2}{1+a^2b^2} + \tan^{-1} \frac{b^2-c^2}{1+b^2c^2} + \tan^{-1} \frac{c^2-a^2}{1+c^2a^2}$

14.
$$\sec^2(\tan^{-1} 2) + \csc^2(\cot^{-1} 3) = 15$$
.

15.
$$\cot^{-1}(\tan 2x) + \cot^{-1}(-\tan 3x) = x$$
.

16.
$$\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{15} + \sin^{-1} \frac{16}{65} = \frac{1}{3}\pi$$
. [C. U. 1941]

17.
$$4(\cot^{-1} 3 + \csc^{-1} \sqrt{5}) = \pi$$
. [C. U. 1939]

18. If
$$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$$
, show that $x + y + z = xyz$.

19. If
$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z - \frac{1}{2}\pi$$
, show that $yz + zx + xy = 1$.

20. If
$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$$
, show that $x^2 + y^2 + z^2 + 2xyz = 1$.

21. If
$$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$$
, show that $x \sqrt{1-x^2} + y \sqrt{1-y^2} + z \sqrt{1-z^2} = 2xyz$.

22. Find the values of

(i)
$$\sin (\sin^{-1} \frac{1}{2} + \cos^{-1} \frac{1}{2})$$
. [C. U. 1935]

(ii) $\cot (\tan^{-1} a + \cot^{-1} a)$.

(iii)
$$\tan \left(\frac{1}{3} \sin^{-1} \frac{2x}{1+x^2} + \frac{1}{3} \cos^{-1} \frac{1-y^2}{1+y^2} \right)$$

- 23. If $\tan^{-1}y = 4 \tan^{-1}x$, find y as an algebraic function of x.
- 24. If $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in A.P., find out the algebraic relation between x, y, z. If in addition, x, y, z are also in A.P., prove that x=y=z. $\{y \neq 0, 1 \text{ or } -1\}$
 - 25. Solve the following equations:
 - (i) $\tan^{-1} (x+1) + \tan^{-1} (x-1) = \tan^{-1} \frac{8}{31}$.
 - (ii) $\tan^{-1} \frac{2x}{1-x^2} = \sin^{-1} \frac{2a}{1+a^2} \cos^{-1} \frac{1-b^2}{1+b^2}$
 - (iii) $\tan(\cos^{-1}x) = \sin(\tan^{-1}2)$.
 - (iv) $\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x$.
 - (v) $\tan^{-1} \frac{x-1}{x+1} + \tan^{-1} \frac{2x-1}{2x+1} = \tan^{-1} \frac{23}{36}$
 - (vi) $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$
 - (vii) $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$.
 - (viii) $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$.
 - (ix) $\tan^{-1} \frac{2x}{1-x^2} + \cot^{-1} \frac{1-x^2}{2x} = \frac{\pi}{3}$
 - (x) $\cot^{-1}(x-1) + \cot^{-1}(x-2) + \cot^{-1}(x-3) = 0$.
 - 26. Show that
 - (i) $\cot^{-1} \frac{xy+1}{x-y} + \cot^{-1} \frac{yz+1}{y-z} + \cot^{-1} \frac{zx+1}{z-x} = 0$.
 - (ii) $\tan (\tan^{-1}x + \tan^{-1}y + \tan^{-1}z)$ = $\cot (\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$.
 - (iii) $\tan^{-1}(\cot x) + \cot^{-1}(\tan x) = \pi 2x$.

Miscellaneous Examples I

- 1. If $3 \sin \theta + 4 \cos \theta = 5$, show that $\tan \theta = \frac{3}{4}$.
- 2. If $a^2 \sec^2 x b^2 \tan^2 x = c^2$, find cosec x.
- 3. If $x = r \cos \theta \cos \phi$, $y = r \cos \theta \sin \phi$, $z = r \sin \theta$, show that $x^2 + y^2 + z^2 = r^2$.
 - 4. If $\sin \theta = \frac{x-y}{x+y}$, show that $\tan \left(\frac{\pi}{4} \frac{\theta}{2}\right) = \pm \sqrt{\frac{y}{x}}$.
 - 5. If $x = r \sin (\theta + 45^{\circ})$ and $y = r \sin (\theta 45^{\circ})$, then $x^2 + y^2 = r^2$.
 - 6. If $\cos (\alpha + \beta) \sin (\gamma + \theta) = \cos (\alpha \beta) \sin (\gamma \theta)$, then $\tan \theta = \tan \alpha \tan \beta \tan \gamma$.

Show that (Ex. 7 to 9) :--

- 7. $(\cos x \cos y)^2 + (\sin x \sin y)^2 = 4 \sin^2 \frac{x y}{2}$
- 8. $\sin A + \sin B + \sin C \sin (A + B + C)$ = $4 \sin \frac{A+B}{2} \sin \frac{B+C}{2} \sin \frac{C+A}{2}$.
- 9. $4 \sin \frac{A+B+C}{2} \sin \frac{B+C-A}{2} \sin \frac{C+A-B}{2} \sin \frac{A+B-C}{2}$ = $1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C$.
- 10. If $\tan \beta = \frac{2 \sin \alpha \sin \gamma}{\sin (\alpha + \gamma)}$, then $\tan \alpha$, $\tan \beta$, $\tan \gamma$ are in harmonical progression.
 - 11. If $a + \beta + \gamma = (2n + 1) \frac{n}{2}$, then
 - (i) $\tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1$.
 - (ii) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = \pm 4 \cos \alpha \cos \beta \cos \gamma$.
 - 12. If the angles A, B, C be in A.P., then $\frac{\sin A \sin C}{\cos C \cos A} = \frac{\cos B}{\sin B}.$

- 13. If cosec $2A + \operatorname{cosec} 2B + \operatorname{cosec} 2C = 0$, show that $\tan A + \tan B + \tan C + \cot A + \cot B + \cot C = 0$.
 - 14. If $\tan \alpha = \frac{a \sin \beta}{1 a \cos \beta}$ and $\tan \beta = \frac{b \sin \alpha}{1 b \cos \alpha}$

then $\frac{\sin a}{\sin \beta} = \frac{a}{b}$.

15. Show that

 $\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta = \cot \theta$.

- 16. If $\cos (\theta \psi) \cos \phi = \cos (\theta \phi + \psi)$, then $\tan \theta$, $\tan \phi$, $\tan \psi$ are in harmonical progression.
- 17. If $1 + \cos(y-z) + \cos(z-x) + \cos(x-y) = 0$, show that either (y-z), or (z-x), or (x-y) is an odd multiple of π .
 - 18. If $\sin \theta + \sin \phi = \sqrt{3} (\cos \phi \cos \theta)$, show that $\sin 3\theta + \sin 3\phi = 0$.
 - 19. Eliminate a and β from $\sin \alpha + \sin \beta = a$, $\cos \alpha + \cos \beta = b$, $\cos (\alpha \beta) = c$.
 - 20. If $A+B+C=\pi$, prove that
 - (i) $\tan B \tan C + \tan C \tan A + \tan A \tan B$ = 1 + $\sec A \sec B \sec C$.
 - (ii) $\cot A + \cot B + \cot C = \cot A \cot B \cot C$ + $\csc A \csc B \csc C$.
 - 21. If $A + B + C = \pi$, and if $\sin^2 A + \sin^2 B + \sin^2 C = \sin B \sin C + \sin C \sin A + \sin A \sin B$, then A = B = C,
- 22. If A, B, C be the angles of a triangle, and if cot A + cot B + cot $C = \sqrt{3}$, show that the triangle is equilateral.
- 23. If $\sec ax + \sec bx = 0$, show that the values of x form two series in A.P.

CHAPTER XIII

LOGARITHMS

72. Definition of Logarithm.

Logarithm of a number with respect to a given base is the index of the power to which the base is to be raised in order to give the number.

Mathematically, if $a^x = N$, then 'x' is the index of the power to which 'a' (which is called the base) is raised to give 'N'. Hence, by definition, 'x' is the logarithm of 'N' with respect to the base 'a' and it is usually written as $x = log_a N$.

As a numerical example, $\log_2 8=3$, for $2^8=8$ *i.e.*, 3 is the power to which 2 is to be raised to give 8. Again, since $3^4=81$, $4=\log_3 81$.

Any result involving indices can be expressed as a result in logarithm, and vice versa.

For example,

if
$$p^a = r$$
, then, $q = \log_p r$,
if $m^n = z^k$, then $n = \log_m (z^k)$,
or, $k = \log_z (m^n)$.

Similarly, if
$$\log_y x = z$$
,
then $y^z = x$.

It should be noted that the logarithms of the same number with respect to different bases will be different; for example, to get the same number 64, we must raise 2 to the power 6, whereas we are to raise 4 to the power 3 and 8 to the power 2 only; hence $\log_2 64 = 6$, $\log_4 64 = 3$, $\log_8 64 = 2$.

Thus, so long as the base is not stated, logarithm of a number has no meaning.

73. Special Cases.

We know from Algebra that if a be any real finite quantity, other than zero, then $a^0 = 1$.

Hence, $\log_a 1 = 0$; in other words.

(i) logarithm of 1 with respect to any finite quantity (other than zero) as base, is zero.

Again, a being any quantity, $a^1 = a$.

Hence, $1 = \log_a a$; in other words,

(ii) logarithm of any number with respect to itself as base is unity.

Note 1. If
$$a^x = 0$$
, then $x = -\infty$ if $a > 1$, and $x = +\infty$ if $a < 1$.

Thus, we have $\log_a 0 = \mp \infty$ according as a > 0 or < 1. Hence, logarithm of zero to a base greater than unity is minus infinity, and to a base less than unity is plus infinity.

Note 2 Since the equation $a^2 = -n$ (a and n being real positive quantities), cannot be satisfied by any real value of x, whether positive or negative, provided we consider the principal value only of a^2 , therefore, logarithm of a negative quantity (in a system of logarithms whose base is a real positive quantity) must be imaginary.

74. Fundamental formulæ in logarithms.

From the definition it is clear that logarithms are but indices in another form. Hence, corresponding to the three fundamental results in the theory of indices in Algebra, namely that if a, x, y be any real quantities,

(i)
$$a^x \times a^y = a^{x+y}$$
.

(ii)
$$a^x + a^y = a^{x-y}$$
, and

(iii)
$$(a^x)^y = a^{xy}$$
,

we get three fundamental laws of logarithms which are given below.

^{*} See a treatise on Higher Trigonometry.

(i) $\log_a (m \times n) = \log_a m + \log_a n$.

In other words, logarithm of the product of two quantities is equal to the sum of their logarithms taken separately, base remaining the same always.

Proof. Put
$$\log_a m = x$$
, $\log_a n = y$ and $\log_a (m \times n) = z$.

Then from definition,

$$a^x = m$$
, $a^y = n$ and $a^z = m \times n = a^x \times a^y = a^{x+y}$,

so that, z = x + y.

Replacing the values,

$$\log_a(mn) = \log_a m + \log_a n.$$

Cor. $\log_a(m.n.p...) = \log_a m + \log_a n + \log_a p + \cdots$

(ii)
$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$
.

In other words, logarithm of the quotient of two numbers is equal to the difference of their logarithms (logarithm of the numerator minus logarithm of the denominator).

Proof. Put
$$\log_a m = x$$
, $\log_a n = y$ and $\log_a \binom{m}{n} = z$.

Then, from definition,

$$a^x = m$$
, $a^y = n$
and $a^x = \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$,

so that

$$z = x - y$$

or replacing the values,

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

(iii) $\log_a (m)^n = n \log_a m$.

Or, logarithm of a power of a number is the product of the power and the logarithm of the number.

Proof. Put $\log_a m = x$, and $\log_a (m)^n = z$.

Then, by definition,

$$a^x = m$$
 and $a^z = (m)^n = a^{nx}$.
 \vdots $z = nx$.

or replacing the values,

$$\log_a (m)^n = n \log_a m.$$

Ex. 1. Reduce to a simple form $\log_a \frac{x^p y^q}{z^q}$.

$$\log_a \frac{x^p y^q}{z^s} = \log_a (x^p y^q) - \log_a (z^s)$$

$$= \log_a x^p + \log_a y^q - \log_a z^s$$

$$= p \log_a x + q \log_a y - s \log_a z.$$

Ex. 2. Simplify log 10 3/28.

$$\log_{10} \sqrt[3]{25} = \log_{10} \left(\frac{5^2}{8.11}\right)^{\frac{1}{3}} = \frac{1}{3} \log_{10} \frac{5^2}{2^8.11}$$

$$= \frac{1}{3} \log_{10} \frac{10^2}{2^5.11}$$

$$= \frac{1}{3} \left[\log_{10} 10^2 - \log_{10} (2^5.11)\right]$$

$$= \frac{1}{3} \left[2 \log_{10} 10 - (\log_{10} 2^5 + \log_{10} 11)\right]$$

$$= \frac{1}{4} \left[2 - 5 \log_{10} 2 - \log_{10} 11\right].$$

75. Change of base.

There is a fourth standard formula whereby logarithms of numbers with respect to one base being given, those with respect to a different base may be obtained. The formula is $\log_a m = \log_b m \times \log_a b$.

Proof. Put $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$. Then, from definition,

$$a^{x} = m, b^{y} = m, a^{y} = b.$$

Hence,
$$x^{\alpha} = m = b^{y} = (a^{z})^{y} = a^{yz}$$
,

or.
$$x = yz$$
.

Replacing the values,

$$\log_a m = \log_b m \times \log_a b.$$

Cor. 1. In the above result, put m=a. Then remembering that $\log_a a=1$, we get

$$log_b a \times log_a b = 1$$
.

Since, the above relation is very important, we add here an idependent proof of it.

Let $\log_b a = x$, and $\log_a b = y$.

Then, $b^x = a$ and $a^y = b$.

$$a = b^{x} = (a^{y})^{x} = a^{xy}, \quad \therefore \quad xy = 1.$$

i.e., $\log_b a \times \log_a b = 1$,

or,
$$\log_b a = \frac{1}{\log_a b}$$
.

Cor. 2. The result of the above article may be written with the help of Cor. 1, in the form

Thus, if logarithms of both m and a with respect to b be known, logarithm of m with respect to a is obtained.

76. Common system of logarithms.

For all practical purposes wherever logarithms are used for numerical calculations, the base is usually taken as 10. Logarithms of numbers with respect to the base 10 are referred to as the Common system of logarithms. The advantage of the common system of logarithms for practical applications will be clear presently, from the Article 77, Theorems I & II.

Note. In higher mathematics, for theoretical investigations, another quantity 'e' (defined in books of Algebra), whose value is nearly 2.718... is used as the base of logarithms, and logarithms to this base e are called Navierian logarithms.

With the help of the logarithmic series established in books on Algebra, Napierian logarithms of numbers are tabulated. The factor $\frac{1}{\log_e 10}$ which is known as the modulus of the common system, applied to the Napierian logarithms will convert them to common logarithms (See Art. 75). Thus, a table of common logarithms is prepared.

Henceforth, we shall proceed with the consideration of the common system of logarithms, and the base being understood to be 10, will not be written.

77. Characteristic and Mantissa of common logarithms.

It is only in very few cases that the logarithm of a number is integral. In most cases, however, the logarithm of a number is partly integral and partly fractional (or decimal).

Def. The integral portion of the logarithm of a number is called the *characteristic*, and the decimal portion is called the *mantissa*.

In case the logarithm of a number is negative, and partly integral and partly decimal, the decimal portion, i.e., the mantissa is always kept positive by altering the integral part, i.e., the characteristic suitably. Thus, the mantissa part of the logarithm of a number is always positive. For instance, if the logarithm of a number is -2.3, we write it as -3+.7 and call -3 as the characteristic and .7 (and not -3) as the mantissa. -3+.7 is often abbreviated in the form 3.7.

Theorem I. The characteristic of the common logarithm of (i) any number greater than 1 is positive, and numerically one less than the number of digits in the integral part of the quantity whose logarithm is sought; and (ii) of any positive* number less than 1, is negative, and numerically one greater than the number of zeros immediately after the decimal point in the quantity whose logarithm is wanted.

(i) Let the number be greater than unity.

Any number, say 7'209, which consists of 1 digit only in its integral part, lies between 1 and 10.

Now, $10^{\circ} = 1$ and $10^{1} = 10$.

Hence, if $10^x = 7.209$, clearly x must be greater than 0 and less than 1.

Thus, log 7'209 must be between 0 and 1, i.e., of the form 0'..., having its characteristic 0.

Similarly, numbers of the type 53'0528, which consists of 2 digits in their integral parts, must lie between 10 and 100 i.e., between 10¹ and 10².

Hence, the index to which 10 should be raised to give 53'0528 must be greater than 1 and less than 2, i.e., log 53'0528 must be of the form 1'... having the characteristic 1.

log 10 is 1, and 10 also falls in this category of two digits.

In the same way, a number which has n digits in its integral part lies between 10^{n-1} (which also has n digits) and 10^n (which has n+1 digits). Thus, the logarithms of such numbers must lie between n-1 and n, i.e., (n-1) + some positive proper fraction. Hence, the characteristic in such cases is n-1.

Hence, the result.

^{*}Logarithms of negative numbers are easily seen to be imaginary, for there is no real power, positive or negative, to which 10 may be raised to give a negative result. [See Note 2, Art. 73]

(ii) Let the number be positive, and less than 1 (i.e., between 0 and 1).

We notice that

$$10^{\circ} = 1$$

$$10^{-1} = \frac{1}{10} = 1$$

$$10^{-2} = \frac{1}{100} = 01$$

$$10^{-3} = \frac{1}{1000} = 001$$

$$10^{-4} = \frac{1}{10000} = 0001$$
etc. etc. etc.

Now, a number less than 1, with no zero immediately after the decimal point, like 3015, must be greater than 1 and less than 1; hence, the power to which 10 must be raised to give such a number must lie between -1 and 0, i.e., =-1+ a positive proper fraction. Hence, such numbers have the characteristic of their logarithms =-1.

A decimal number with one zero immediately after the decimal point, like '078005, lies between '01 and '1 which are respectively equal to 10^{-2} and 10^{-1} .

Hence, if $10^x = 078005$, x must lie between -1 and -2 i.e., x is of the form -1 ······ Writing the decimal part of x positively, in the form $-2 + \cdots$, we notice that the integral part of x, i.e., the characteristic of the logarithm of 078005 is -2.

Similarly, the logarithms of numbers between '01 and '001 (i.e., 10^{-2} and 10^{-3}) which must have two zeros after the decimal point, lie between -2 and -3, i.e., are of the form $-2 \cdot \cdot \cdot \cdot = -3 + \cdot \cdot \cdot \cdot$, and so the characteristic in such cases is -3.

and so on.

Hence the result.

Theorem II. All numbers, formed of the same digits in the same order, differing only in the positions of their decimal points, have the mantissæ of their loyarithms same.

This will be clear from an example. Let us take the numbers 835107, 835107000, 835107, '835107, '000835107 and 8351'07.

Now,
$$\log 835107000 = \log (835107 \times 1000)$$

 $= \log 835107 + \log 1000$
 $= \log 835107 + 3.$
Again, $\log 83'5107 = \log \frac{835107}{10000}$
 $= \log 835107 - \log 10000$
 $= \log 835107 - 4.$
 $\log '835107 = \log \frac{835107}{1000000} = \log 835107 - 6.$
 $\log '000835107 = \log \frac{835107}{10^9} = \log 835107 - 9.$
 $\log 8351'07 = \log \frac{835107}{10^9} = \log 835107 - 2.$

Thus, the logarithms of all the numbers here differ from the logarithm of 835107 by a whole number in each case and so must have their decimal parts, i.e., their mantissæ the same as that of log 835107.

In fact, numbers formed of the same digits in the same order differing only in the position of their decimal points, must have their ratios equal to an integral power of 10 and so must have their logarithms differing only by a whole number.

Hence the result.

The two theorems above given show that (i) the characteristic of the logarithm of a number can be found by a simple glance at the number and (ii) that for the mantissa part of the logarithm of a number, we need only take into

account the digits of which the number is formed, without taking any notice of the position of the decimal point in it.

In logarithmic tables, only the mantissm of the logarithms of numbers are therefore given.

These constitute the special advantages of the common system of logarithms.

78. Examples worked out.

$$\log \sqrt[4]{5}$$
. $\sqrt{10}/2$, and find its value, given $\log 2 = 30103$ and $\log 3 = 4771213$.

The given exp. =
$$\log \frac{5^{\frac{1}{4}} \cdot 2^{\frac{1}{10}}}{(18 \cdot 2^{\frac{1}{2}})^{\frac{1}{3}}}$$

$$- \log \frac{10^{\frac{1}{4}} \cdot 2^{\frac{1}{10}}}{2^{\frac{1}{4}}(2 \cdot 3^{2} \cdot 2^{\frac{1}{2}})^{\frac{1}{3}}} = \log \frac{10^{\frac{1}{4}} \cdot 2^{\frac{1}{10}}}{2^{\frac{1}{4}} \cdot 2^{\frac{1}{3}}}$$

$$= \log \frac{10^{\frac{1}{4}}}{2^{\frac{1}{30}} \cdot 3^{\frac{1}{3}}} = \log 10^{\frac{1}{4}} - \log (2^{\frac{1}{30}} \times 3^{\frac{2}{3}})$$

$$= \frac{1}{4} \log 10 - (\log 2^{\frac{1}{30}} + \log 3^{\frac{2}{3}})$$

$$= \frac{1}{4} \log 10 - \frac{1}{10} \log 2 - \frac{2}{3} \log 3$$

and its value is

$$\frac{1}{4} \cdot 1 - \frac{1}{2} \frac{8}{6} (30103) - \frac{2}{3} (4771213)$$

$$= 25 - 1956695 - 3180809$$

$$= -1 + 7362496$$

$$= 77362496.$$

Note. $\log 5 = \log \frac{1}{3} = \log 10 - \log 2 = 1 - \log 2$ and hence $\log 5$ is deducible from $\log 2$.

$$7 \log \frac{10}{9} - 2 \log \frac{25}{12} + 3 \log \frac{81}{10} = \log 2$$
.

The left-hand expression

$$= \log \left(\frac{10}{9}\right)^7 - \log \left(\frac{25}{24}\right)^2 + \log \left(\frac{81}{80}\right)^8$$

$$= \log \frac{\left(\frac{10}{9}\right)^7 \times \left(\frac{81}{80}\right)^8}{\left(\frac{25}{4}\right)^2}$$

$$= \log \left\{ \left(\frac{10}{3^2}\right)^7 \times \left(\frac{3^4}{10 \times 2^3}\right)^3 \times \left(\frac{3 \times 2^8 \times 2^2}{10^2}\right)^2 \right\}$$

$$= \log \left(\frac{10^7}{3^{14}} \times \frac{3^{12}}{10^8 \times 2^9} \times \frac{3^2 \times 2^{10}}{10^4}\right)$$

$$= \log 2.$$

Alternative method:

Left side

$$= 7(\log 10 - \log 9) - 2(\log 25 - \log 24) + 3(\log 81 - \log 80)$$

$$= \{\log (5 \times 2) - \log 3^2\} - 2\{\log 5^2 - \log (3 \times 2^3)\}$$

$$+ 3\{\log 3^4 - \log (5 \times 2^4)\}$$

$$= 7\{\log 5 + \log 2 - 2 \log 3\} - 2\{2 \log 5 - \log 3 - 3 \log 2\} + 3\{4 \log 3 - \log 5 - 4 \log 2\}$$

 $= \log 2$.

Ex. 3. Find the number of digits in 4^{15} , having given $\log 2 = 30103$.

We have

$$\log 4^{15} = \log 2^{80} = 30 \log 2$$

= 30 × '30103 = 9'0309.

Hence, since the characteristic of log 4¹⁵ is 9, 4¹⁵ must consist of 10 digits.

Ex. 4. Find approximately the 7^{th} root of 3.528, having given $\log 2 = 30103$, $\log 3 = 4771213$. $\log 7 = 8450980$ and $\log 1197.342 = 3.0782184$.

Let
$$x = (3.528)^{\frac{1}{7}} = \left(\frac{7^2 \times 3^2 \times 2^8}{10^3}\right)^{\frac{1}{7}}$$

then $\log x = \frac{1}{7} \left[2 \log 7 + 2 \log 3 + 3 \log 2 - 3 \log 10\right]$
 $= \frac{1}{7} \left[2 \times 8450980 + 2 \times 4771213 + 3 \times 30103 - 3\right]$
 $= 0782184$ nearly.

Now. log 1197'342 = 3'0782184.

... log 1'197342 = '0782184. having characteristic 0, but mantissa same as that of log 1197'342.

Hence, x = 1.197342 approximately.

Ex. 5. Obtain an approximate numerical solution of 2^{x} . $3^{2x} = 100$. having given log 2 = 30103, log 3 = 47712.

We have

$$2^{x} \cdot 3^{2x} = 10^{2}$$
.

 $\log (2^x, 3^{2x}) = \log 10^2$

i.e., $x \log 2 + 2x \log 3 = 2 \log 10 = 2$.

$$x = \frac{2}{\log 2 + 2 \log 3} = \frac{2}{30103 + 2 \times 47712}$$
= 1.5933 nearly.

Note. Equations of this type are called Exponential Equations.

Examples XIII(a)

[Use the values: log 2 = 30103, log 3 = 4771213. log 7 = '8450980 when required]

- 1. Find the logarithm of (i) 1728 to the base $2\sqrt{3}$. (ii) cos⁸ a to the base sec a.
 - 2. Find log. 10000.
 - Show that $\log_{10} 2$ lies between 1 and 1.

[C. U. 1926]

- 4. Prove that
 - (i) $\log_a m \times \log_b n = \log_b m \times \log_a n$.
 - (ii) $\log_2 \log_2 \log_2 16 = 1$.
- 5. If $\log_e m + \log_e n = \log_e (m + n)$, find m as a simple function of n.
- 6. Prove that if a series of numbers be in G.P., their logarithms are in A.P.

7. Prove that

$$2 \log a + 2 \log a^2 + 2 \log a^3 + \dots + 2 \log a^n$$

= $n(n+1) \log a$.

- 8. If x is positive and less than unity, show that $\log (1+x) + \log (1+x^2) + \log (1+x^4) + \log (1+x^8) + \cdots$ to $\infty = -\log (1-x)$.
 - 9. Simplify
 - (i) $\log_2 \sqrt{6} + \log_2 \sqrt{\frac{2}{8}}$.

(ii)
$$\frac{\log \sqrt{27 + \log 8 - \log \sqrt{1000}}}{\log 1.2}$$

- 10. Find $\log (00225)^{\frac{1}{3}}$ and $\log (\sqrt[6]{2})^{-\frac{1}{3}}$.
- 11. Prove that
 - (i) $\log_a b \times \log_b c \times \log_c a = 1$.
 - (ii) $\log_a x = \log_b x \times \log_a b \times \log_a c \cdots \times \log_n m \times \log_a n$.
- 12. Show that
 - (i) $7 \log \frac{16}{18} + 5 \log \frac{25}{32} + 3 \log \frac{81}{80} = \log 2$.
 - (ii) $7 \log \frac{15}{18} + 6 \log \frac{8}{8} + 5 \log \frac{2}{8} + \log \frac{32}{48} = \log 3$.
- 13. Extract the fifth root of 84, having given log 2425805 = 6.3848559.
- 14. Calculate $(0020736)^{\frac{1}{7}}$, having given $\log 41369 = 4.6166750$.
- 15. Simplify

(i)
$$\log \sqrt[7]{\frac{8^{\frac{1}{8}} \times 14^{\frac{1}{8}}}{\sqrt{72} \times \sqrt[5]{60}}}$$

(ii)
$$\sqrt[8]{\frac{7\cdot2\times6\cdot3}{62\cdot5}}$$
, having given log 898665 = 5.9535977.

- 16. Find the value of $64 \{1 (1.05)^{-20}\}$, having given $\log 24121 = 4.382394$.
- 17. Find the number of digits in (i) 2^{40} , (ii) 3^{11} , (iii) $(540)^9$.
- 18. Find the number of zeros after the decimal point before the first significant digit in the expressions

(i)
$$(024)^{15}$$
. (ii) $\left(\frac{1}{4.05}\right)^{8}$. (iii) $\left(0259\right)^{50}$.

- 19. Solve the equations:-
 - (i) $3^x = 2$. (ii) $3^{x-4} = 7$.
 - (iii) $5^{6x} 7^{x+2} = 3^{2x-3}$.

- 20. (i) If $\log (x^2y^3) = a$, $\log {x \choose y} = b$, find $\log x$ and $\log y$.
 - (ii) If $a^2 + b^2 = 7ab$, show that $\log \{\frac{1}{3}(a+b)\} = \frac{1}{2}(\log a + \log b)$.
- 21. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, show that $x^x y^y z^z = 1$.
- 22. Why is $\log (1+2+3) = \log 1 + \log 2 + \log 3$?
- 23. If a, b, c,... be in G.P., show that $\log_a x$, $\log_b x$, $\log_c x$ are in H.P.
- 24. If $xy^{l-1} = a$, $xy^{m-1} = b$, $xy^{n-1} = c$, prove that $(m-n) \log a + (n-l) \log b + (l-m) \log c = 0$.
- 25. If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$, show that $y^z z^y = z^x x^z = x^y y^x$.

79. Tables of Logarithms and Trigonometrical ratios.

Several mathematical tables correct up to five places of decimals are given at the end of the book. An explanation of the tables is given below.

Table I gives the common logarithm of all numbers from 1 to 10000, i.e., those which consist of 4 digits or less. The tabulated quantities are the mantisse only, correct to five places, with the decimal point dropped. The characteristic is to be supplied according to the rule given in Art. 77. The main body of the table gives logarithms (mantissa part) of numbers of 3 digits, and the mean difference table at the side supplies the increment in the mantissa due to the fourth digit. This increment is written, in order to save space, giving the significant digits only, which are to be supplied with the necessary number of zeros to make up 5 places (here the table being a five-figure table). Thus, '00024 will be written as 24 only in the difference table. As an example, to find log 2'697, we notice from the table that the mantissa for log 269 is 42975, and along the same row, the difference table gives 115 under the heading 7. This means that for 7 in the fourth place of the number (i.e., for the number 2697) the increment in the mantissa will be '00115. Hence, log 2697 will have its mantissa '42975 + '00115 = '43090. Again, log 2'697 has the same but its characteristic is 0. Thus, log 2'697 mantissa = 0.43090.

Table II gives ordinary sines and cosines (usually referred to as natural sines and cosines) of all angles from 0° to 90° at intervals of 1′, sines being given from the left side of the top towards the right and downwards, and cosines being given from the right side of the bottom towards the left and upwards. The table is arranged in such a way that the sine of any angle given is the same as the cosine of exactly the complementary angle, and it is on this arrangement that a single table serves as a sine as well as a cosine table. The main portion of the table gives sines or cosines of angles at intervals of 10′, and the difference

table at the side gives changes in the value of the sine or cosine for changes in minutes in the angles. It should be remembered that as an angle increases from 0° to 90°, its sine increases from 0 to 1 whereas its cosine decreases from 1 to 0. Hence, the changes given in the difference table are to be added in case of sines and subtracted in case of cosines for the increased number of minutes in the angles. Moreover, as in Table I, the numbers in the difference table are to be made up to five places of decimals by supplying the requisite number of zeros before it. For example, using the table, sin 53° 23′ = '80212 + '00052 = '80264 and cos 20° 42′ = '86892 - '00029 = '86863.

Table III similarly gives natural tangents and cotangents of angles from 0° to 90°, obtained at intervals of 1' with the help of the difference table. The quantities in the difference table, being made up into five figures, are to be added in case of tangents and subtracted in case of cotangents for increased number of minutes in the angle.

Table IV gives logarithmic sines and logarithmic cosines of all angles from 0° to 90° at intervals of 1' (with the aid of the difference table). Logarithmic sine of angle θ , written as L sin θ means $10 + \log \sin \theta$, and similarly, logarithmic cosine of θ , written as L cos θ means $10 + \log \cos \theta$. In taking logarithms of trigonometrical ratios of angles, it may be noted that sines and cosines of angles are numerically less than unity, and tangents of angles between 0° and 45° as also cotangents of angles between 45° and 90° are less than unity. Hence, logarithms of these quantities are negative. To avoid using negative values in the tables, logarithms of trigonometrical ratios are always tabulated after adding 10 to them. Thus, the table gives L sin θ and L cos θ (and not log sin θ and log cos θ).

Table V gives logarithmic tangents (i.e., $L \tan \theta = 10 + \log \tan \theta$) and logarithmic cotangents (i.e., $L \cot \theta = 10 + \log \cot \theta$) of all angles from 0° to 90°, obtained at intervals of 1' with the aid of the difference table.

80. Principle of Proportional Parts.

Suppose we find from table I the logarithms of the two numbers 6257 and 6258, and we want to find the logarithm of 62576; or that we find from table III, tan 53° 23' and tan 53° 24', but we want to find tan 53° 23' 20"; or similarly, from table IV, we get $L\cos 37^{\circ}42'$ and $L\cos 37^{\circ}43'$ but we want to find $L\cos 37^{\circ}42'$ 48"; how are we to proceed?

In order to meet such cases, the 'Principle of Proportional Parts' may be used. The principle may be stated as follows:

If the value of a quantity depending on a variable quantity x be tabulated for different values of x at regular small intervals, then in most cases, for a very small change in x (which is called the argument) the corresponding small change in the tabulated quantity (called the function of the argument) is proportional to the change in x.

We shall assume the truth of this principle; for a strict proof of it, with the proper restriction under which it is true, depends on the use of Calculus. For the tables with which we are concerned, it is true for all practical purposes.

The application of the principle is illustrated in the following examples:

Ex. 1. Given $\log 63374 = 4.8019111$ and $\log 63375 = 4.8019180$, find $\log 63.3743$ and find the number whose logarithm is $\overline{2.8019136}$.

Here, $\log 63375 = 4.8019180$

and $\log 63374 = 4'8019111$.

Hence, for an increase of 1 in the number, the increment in the logarithm is '0000069. (This is usually spoken as 'diff. for 1 is 69')

Therefore, by the Principle of Proportional Parts, increase in the logarithm for an increase of '3 in the number is

$$3 \times 0000069 = 00000207$$

= '0000021, up to seven places.

Hence, $\log 63374^{\circ}3 = 4^{\circ}8019111 + {^{\circ}0000021}$ = $4^{\circ}8019132$.

 $\log 63^{\circ}3743 = 1^{\circ}8019132.$

Again, 4'8019136 lies between 4'8019111 and 4'8019180, the difference from the former being '0000025. Hence, 4'8019136 is the logarithm of a number lying between 63374 and 63375, say logarithm of 63374 + x.

Then, diff. for 1 being 69 (i.e., '0000069) and diff. for x being 25, (i.e., '0000025), by the Principle of Proportional Parts, we have

or,
$$x = \frac{25}{60} = 36 \cdots$$

Hence, $\log 6337436\dots = 48019136$.

The required number whose logarithm is $\overline{2}$:8019136, having the same mantissa, must be formed of the same digits arranged in the same order, and its characteristic being -2, the number must be '06337436...

Ex. 2. (i) Given
$$L \sin 37^{\circ} 43' 50'' = 9.7867152$$

 $L \sin 37^{\circ} 44' = 9.7867424$,

find L sin 37° 43' 56".

(ii) Given L tan 79° 51′ 40″ = 10.7475657L tan 79° 51′ 50″ = 10.7476872,

find the angle whose L tan is 10'7476532.

In (i) diff. (in the value of $L \sin$) for 10" (diff. in angle) = 272 (i.e., '0000272)

hence, diff. for $6'' = \frac{6}{10} \times 272 = 163^{\circ}2$ *i.e.*, '00001632 and so $L \sin 37^{\circ} 43' 56'' = 9.7867152 + '0000163 = 9.7867315.$

In (ii) the angle whose L tan is 10.7476532 evidently lies between 79° 51′ 40″ and 79° 51′ 50″.

Let the angle be 79° 51' 40'' + x''.

Now, diff. (in the value of L tan) for 10" (diff. in angle) = 1215 (i.e., 0001215)

and diff. for x'' = 875

(i.e., 0000875, being 10'7476532 - 10'7475657)

$$\therefore \frac{x}{10} = \frac{875}{1215}$$
 or $x = 7.2$ nearly.

Thus, the required angle is 79° 51′ 47"'2.

Ex. 3. Given $\cos 58^{\circ} 17' = 5257191$ and diff. for 1' = 2474, find $\cos 58^{\circ} 17' 20''$.

Here, diff. for 1' i.e., 60'' = 2474,

 \therefore diff. for $20'' = \frac{20}{50} \times 2474 = 825$ (nearly).

As for increasing angle, cosine diminishes,

$$3.5 \cos 58^{\circ} 17' 20'' = 5257191 - 0000825$$

= 5256366.

Examples XIII(b)

- 1. Given log 18'906 = 1'2765997 and log 18'907 = 1'2766226, find log 1890'635.
 - 2. Given log 69714 = 4'8433200 log 69715 = 4'8433262,

find $\log (000697145)^{\frac{1}{8}}$.

3. Given $\log 37602 = 4.5752109$ $\log 37601 = 4.5751994$.

find the number whose logarithm is 1.5752086.

4. Given $\log 3 = 4771213$ $\log 74008 = 4.8692787$ diff. for 1' = 59.

find ('09)#.

5. Given $\cos 32^{\circ} 16' = 8455726$ and cos 32° 17' = '8454172. find the value of cos 32° 16′ 24"

and find the angle whose cosine is '8455176.

- 6. Find tan 38° 24′ 37.5″, having given $\tan 38^{\circ} 24' = 7925902$ and $\tan 38^{\circ} 25' - 7930640$.
- 7. Given $L \sin 44^{\circ} 17' = 9.8439842$ and $L \sin 44^{\circ} 18' = 9.8441137$. find L sin 44° 17′ 33″. Deduce the value of L cosec 44° 17′ 33″.
 - 8. Given $L \sin 36^{\circ} 24' = 9.7733614$ $L \sin 36^{\circ} 25' = 9.7735327$.

find the angle whose L sin is 9.7734642.

9. If $L \cot 53^{\circ} 13' = 9.8736937$ $T_{\ell} \cot 53^{\circ} 14' = 9.8734302.$

find θ where L cot $\theta = 9.8734523$.

10. Given L tan $22^{\circ} 37' = 9.6197205$ diff. for 1' = 3557.

find the value of

L tan 22° 37′ 22" and the angle whose L tan is 9.6195283. 11. Prove that, θ being any acute angle. $L \sin \theta + L \csc \theta = L \cos \theta + L \sec \theta$ $-I_{L}$ tan $\theta + I_{L}$ cot $\theta = 20$.

- 12. Given $L \cos 36^{\circ} 40' = 9.9042411$, find $L \sec 36^{\circ} 40'$.
- 13. Given $L \cos 34^{\circ} 44' = 9.9147729$ $L \cos 34^{\circ} 45' = 9.9146852.$

find the value of L cos 34° 44′ 27".

14. Given $L \sin 36^{\circ} 40' = 9.7760897$ $L \cos 36^{\circ} 40' = 9.9042411.$

find L tan 36° 40'.

- 15. Prove that the difference of tabular logarithms of any two ratios is equal to the difference of the logarithms of those two ratios.
 - 16. If $\sin \theta = 8$, find θ . given $\log 2 = 3010300$, $T_{\rm sin}$ 53° 7′ = 9.9030136 $L_{\text{sec }36^{\circ}52'} = 10.0968916.$
 - 17. Find the value of $\sin 34^{\circ} 17' \times \cos 77^{\circ} 23'$ tan 27° 12'

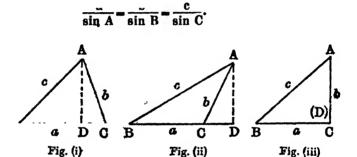
given $L \sin 12^{\circ} 37' = 9.3393$ $L \cos 55^{\circ} 43' = 9.7507$ L tan 62° 48' = 10'2891 $\log 23.94 = 1.3791$

and

CHAPTER XIV

PROPERTIES OF TRIANGLES

- 81. In a triangle ABC, there are six parts, the three sides and the three angles. It is usual to denote the angles of the triangle by A, B, C and the corresponding opposite sides by a, b, c. The six parts are not independent of one another. The various relations existing among them are deduced in the following articles.
 - 82. In any triangle, prove that



Let ABC be any triangle. From A draw AD perpendicular to BC or BC produced if necessary [Fig. (ii)]

[In Fig. (i), C is an acute angle, in Fig. (ii), C is an obtuse angle, in Fig. (iii), C is a right angle.]

From
$$\triangle ABD$$
, $AD = AB \sin ABD = c \sin B$.
From $\triangle ACD$, $AD = AC \sin ACD = b \sin C$ [Fig. (i)]
or, = $b \sin (\pi - C)$ [Fig. (ii)]
i.e., = $b \sin C$.

...
$$b \sin C = c \sin B$$
, i.e., $\sin B \sin C$

Similarly, by drawing a perpendicular from B upon CA,

we have
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

In Fig. (iii), C is a right angle;

$$\sin A = \frac{a}{c} ; \sin B = \frac{b}{c} ; \sin C = 1.$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = c = \frac{c}{\sin C}.$$

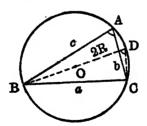
Hence, in all cases,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \cdots \quad (1)$$

Thus, in any triangle,

the sides are proportional to the sines of the opposite angles.

An alternative method of proof.



Let O be the centre and R be the radius of the circle circumscribing the triangle ABC.

Join BO and produce it to meet the circumference in D. Join CD. The $\angle BCD$ is then a right angle.

From
$$\triangle BCD$$
, $\sin BDC = \frac{BC}{BD} = \frac{a}{2R}$.

But $\angle BDC = \angle A$, being in the same segment,

$$\therefore \quad \frac{a}{2R} = \sin A, \quad \text{or,} \quad \frac{a}{\sin A} = 2R.$$

Similarly, by joining AO and producing it to meet the circumference in E, and joining CE, BE, it can be shown that

$$\frac{b}{\sin B} = 2R \text{ and } \frac{c}{\sin C} = 2R.$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \qquad \cdots (2)$$

Note 1. If angle A be obtuse, A and D fall on opposite sides of BC and ABCD being cyclic, $\sin BDC = \sin (180^{\circ} - A) = \sin A$, and the same result follows. In case A is a right angle, evidently $2R = a = a/\sin A$, and we get the same result.

Note 2. It follows from the relation (2) that $a=2R \sin A$, $b=2R \sin B$, $c=2R \sin C$; $\sin A=\frac{a}{2R}$, $\sin B=\frac{b}{2R}$ on $C=\frac{c}{2R}$.

83. In any triangle, to prove that

$$a^{2} = b^{2} + c^{2} - 2bc \cos A, \text{ or, } \cos A = \frac{b^{2} + c^{2} - a^{2}}{2bc}.$$

$$b^{2} = c^{2} + a^{2} - 2ca \cos B, \text{ or, } \cos B = \frac{c^{2} + a^{2} - b^{2}}{2ca}.$$

$$c^{2} = a^{2} + b^{2} - 2ab \cos C, \text{ or, } \cos C = \frac{a^{2} + b^{2} - c^{2}}{2ab}.$$

Take the figures of Art. 82.

First, let C be an acute angle [Fig. (i)]; then from Geometry.

 $AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$

Now, from $\triangle ACD$, $CD = AC \cos C = b \cos C$.

$$c^2 = a^2 + b^2 - 2ab \cos C$$
.

Next, let the angle C be an obtuse angle [Fig. (ii)]; then from Geometry.

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$

Now, from
$$\triangle ACD$$
, $CD = AC \cos ACD$
= $b \cos (\pi - C) = -b \cos C$.

$$c^2 = a^2 + b^2 - 2ab \cos C$$
.

Lastly, let C be a right angle [Fig. (iii)]; then from Geometry,

$$AB^2 = BC^2 + CA^2$$
,
i.e., $c^2 = a^2 + b^2 = a^2 + b^2 - 2ab \cos C$.
 $\therefore \cos C = \cos 90^\circ = 0$.

Hence, for all values of C, we have

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

$$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Similarly, the other two relations can be established.

Obs. This theorem expresses the cosines of the angles of a triangle in terms of the sides.

84. In any triangle, to prove that

 $a = b \cos C + c \cos B$.

 $b = c \cos A + a \cos C$.

c=a cos B+b cos A.

- Take the figures of Art. 82.

In Fig. (i), where C is an acute angle,

$$BC = BD + CD$$

 $=AB\cos ABD+AC\cos ACD.$

$$\therefore a = c \cos B + b \cos C.$$

In Fig. (ii), where C is an obtuse angle,

$$BC = BD - CD$$

 $=AB\cos ABD - AC\cos ACD$

 $= c \cos B - b \cos (180^{\circ} - C).$

 $=c \cos B + b \cos C$.

In Fig. (iii), where C is a right angle,

 $BC = AB \cos B$.

$$\therefore a = c \cos B = c \cos B + b \cos C.$$

$$[\because \cos C = \cos 90^{\circ} = 0.]$$

Thus, in all cases,

$$a = b \cos C + c \cos B$$
.

Similarly, the other two relations can be established.

85. From Art. 83 and note of Art. 82, it follows that

$$\tan A = \frac{\sin A}{\cos B} = \frac{a}{b^2 + c^2 - a^2} = \frac{abc}{R} \cdot \frac{1}{b^2 + c^2 - a^2}$$

Similarly,
$$\tan B = \frac{abc}{R} \cdot \frac{1}{c^2 + a^2 - b^2}$$
.

$$\tan C = \frac{abc}{R} \cdot \frac{1}{a^2 + b^2 - c^2}.$$

86. Trigonometrical ratios of half angles of a triangle in terms of the sides.

We have,
$$2 \sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a - b + c)(a + b - c)}{2bc}$$

Let s denote the semi-perimeter of the triangle;

then
$$2s=a+b+c$$
.

Now,
$$a-b+c=a+b+c-2b=2s-2b=2(s-b)$$
,
 $a+b-c=a+b+c-2c=2s-2c=2(s-c)$.

Hence,
$$2 \sin^2 \frac{A}{2} = \frac{2(s-b) \cdot 2(s-c)}{2bc}$$

i.e.,
$$\sin^{\frac{A}{2}} = \frac{(s-b)(s-c)}{bc}.$$

$$\therefore \sin^{\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

The positive value of the square root must be taken for A, being an angle of a triangle, is less than 180° ; and hence, $\frac{1}{2}A < 90^{\circ}$ and consequently, $\sin \frac{1}{2}A$ must always be positive.

Again,
$$2 \cos^2 \frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$
Now, $b+c-a = a+b+c-2a = 2s-2a = 2(s-a).$

$$\therefore 2 \cos^2 \frac{A}{2} = \frac{2s \cdot 2(s-a)}{2bc}, i.e., \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc}.$$

$$\therefore \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{2bc}}.$$

Here also the positive value of the square root must be taken; for $\frac{1}{2}A$ being less than 90°, $\cos \frac{1}{2}A$ is always positive.

Again,
$$\tan \frac{A}{2} = \sin \frac{A}{2} + \cos \frac{A}{2}$$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} + \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly, the trigonometrical ratios of $\frac{B}{2}$, $\frac{C}{2}$ can be obtained in terms of the sides.

Note. Without assuming the values of $\sin \frac{1}{2}A$, $\cos \frac{1}{2}A$, the value of $\tan \frac{1}{2}A$ can be obtained by substituting the values of $\cos A$ in terms of the sides from Art. 83 in the relation $\tan^2 \frac{1}{2}A = \frac{1-\cos A}{1+\cos A}$ and then extracting the square root after simplification.

Thus, we have

sin
$$\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

sin $\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$... (1)
sin $\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$... (2)
cos $\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$... (2)
cos $\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$... (2)
tan $\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$... (3)
tan $\frac{C}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$... (3)

87. Sine of an angle of a triangle in terms of the sides.

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$= 2 \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}} \cdot [Art. 86]$$

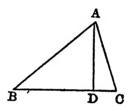
$$\therefore \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$
Similarly, $\sin B = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)}.$

$$\sin C = \frac{2}{cb} \sqrt{s(s-a)(s-b)(s-c)}.$$

 $\sqrt{s(s-a)(s-b)(s-c)}$, being the expression for the area of the triangle [See Art. 88], is usually denoted by the Greek letter \triangle . Hence, the above formulæ may be written as

$$\sin A = \frac{2\triangle}{bc}$$
, $\sin B = \frac{2\triangle}{ca}$, $\sin C = \frac{2\triangle}{ab}$.

88. Area of a triangle.



Let ABC be a triangle and let \triangle denote its area. Draw AD perpendicular to BC; then from $\triangle ACD$,

$$AD = AC \sin C = b \sin C$$
.

Now,
$$\Delta = \frac{1}{2}BC \cdot AD = \frac{1}{2}ab \sin C$$
.

Similarly by drawing perpendiculars from B and C to the opposite sides, it can be shown that

$$\Delta = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$$
.

Otherwise,
$$\triangle = \frac{1}{2}ab \sin C$$

 $= \frac{1}{2}ca \sin B [b \sin C = c \sin B]$
 $= \frac{1}{2}bc \sin A [a \sin B = b \sin A]$

Thus, $\triangle = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$... (i) = $\frac{1}{2}(product \ of \ two \ sides) \times (sine \ of \ included \ angle).$

Again,
$$\triangle = \frac{1}{2}bc \sin A = bc \sin \frac{A}{2} \cos \frac{A}{2}$$

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$
(ii)

Substituting in the expression $s = \frac{1}{2}(a+b+c)$, we get

$$\Delta = \frac{1}{4} \sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$$

$$= \frac{1}{4} \{ 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 \}^{\frac{1}{2}}. \quad \cdots \quad \text{(iii)}$$

Again,

$$\Delta = \frac{1}{2}bc \sin A = \frac{1}{2}bc.\frac{a}{2R} \left[Art. 82 \right] = \frac{\mathbf{abc}}{4R}. \qquad \qquad \cdots \qquad \text{(iv)}$$

Note. In some text books, S is used to denote the area of a triangle, but to avoid confusion between S and s in writing, the symbol \triangle is preferable.

89. In any triangle, to prove that

$$\tan\frac{\mathbf{B}-\mathbf{C}}{2} = \frac{\mathbf{b}-\mathbf{c}}{\mathbf{b}+\mathbf{c}}\cot\frac{\mathbf{A}}{2}.$$

We have, in any triangle,

$$b = \sin B$$
 $c = \sin C$

Similarly,

$$\tan\frac{\mathbf{C}-\mathbf{A}}{2} = \frac{\mathbf{c}-\mathbf{a}}{\mathbf{c}+\mathbf{a}}\cot\frac{\mathbf{B}}{2} \cdot \tan\frac{\mathbf{A}-\mathbf{B}}{2} = \frac{\mathbf{a}-\mathbf{b}}{\mathbf{a}+\mathbf{b}}\cot\frac{\mathbf{C}}{2}$$

90. The three sets of formulæ in Arts. 82, 83, 84 have been established directly from the figures. These three sets

however, are not independent, for, from any one set, the other two sets can be deduced.

For example, let us deduce the formulæ of Art. 83 from those of Art. 84.

By Art. 84,
$$a=b\cos C+c\cos B$$

 $b=c\cos A+a\cos C$
 $c=a\cos B+b\cos A$.

Multiplying these in succession by a, b, and c, and subtracting the first result from the sum of the other two, we have.

$$b^{2} + c^{2} - a^{2} = b(c \cos A + a \cos C) + c(a \cos B + b \cos A)$$

 $-a(b \cos C + c \cos B) = 2bc \cos A.$
 $\therefore \cos A = \frac{b^{2} + c^{2} - a^{2}}{2bc}$; similarly, for $\cos B$, $\cos C$.

Note. For other cases, see Appendix.

91. In working out identities which involve both the sides and angles of a triangle, it is sometimes convenient to express the sides in terms of the angles or the angles in terms of the sides.

Also, it is sometimes found convenient to, express the values of $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ in a form in which the denominator is constant and numerator is free from radical. Thus, multiplying the numerator and the denominator of the value of $\tan\frac{A}{2}$ by $\sqrt{(s-b)(s-c)}$ and noting that

$$\sqrt{s(s-a)(s-b)(s-c)} = \Delta, \text{ we have}$$

$$\tan \frac{A}{2} = \frac{(s-b)(s-c)}{\Delta}; \text{ similarly, } \tan \frac{B}{2} = \frac{(s-c)(s-a)}{\Delta};$$

$$\tan \frac{C}{2} = \frac{(s-a)(s-b)}{\Delta}.$$

Again, multiplying the numerator and the denominator of the value of $\cot \frac{A}{2}$ by $\sqrt{s(s-a)}$ we have

$$\cot\frac{A}{2} = \frac{s(s-a)}{\Delta}.$$

Similarly,
$$\cot \frac{B}{2} = \frac{s(s-b)}{\Lambda}$$
; $\cot \frac{C}{2} = \frac{s(s-c)}{\Lambda}$

Ex. 1. Show that in any triangle,

$$a (\sin B - \sin C) + b (\sin C - \sin A) + c (\sin A - \sin B) = 0.$$

Left side =
$$(a \sin B - b \sin A) + (b \sin C - c \sin B)$$

$$+(c \sin A - a \sin C)$$

= 0 + 0 + 0
$$\left[\because \text{ by Art. 82}, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \right]$$

= 0.

Ex. 2. Show that in any triangle,

$$a \sin (B-C) + b \sin (C-A) + c \sin (A-B) = 0.$$
[H. S. 1961]

$$a = 2R \sin A [by Art. 82] = 2R \sin (B+C),$$

[:: $A+B+C=\pi$]

... 1st term of the left side =
$$2R \sin (B+C) \sin (B-C)$$

= $2R (\sin^2 B - \sin^2 C)$.

Similarly, 2nd term =
$$2R (\sin^2 C - \sin^2 A)$$

3rd term = $2R (\sin^2 A - \sin^2 B)$.

Now adding together the three terms, the required result follows.

Ex. 3. In any triangle, prove that

$$(b-c) \cot \frac{A}{2} + (c-a) \cot \frac{B}{2} + (a-b) \cot \frac{C}{2} = 0.$$

Substituting the values of cot $\frac{A}{2}$, cot $\frac{B}{2}$, cot $\frac{C}{2}$, as given in Art. 91, we have, the left side

$$= (b-c) \cdot \frac{s \cdot (s-a)}{\triangle} + (c-a) \cdot \frac{s \cdot (s-b)}{\triangle} + (a-b) \cdot \frac{s \cdot (s-c)}{\triangle}$$

$$= \frac{s}{\triangle} \left[(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c) \right]$$

$$= \frac{s}{\triangle} \cdot 0 = 0.$$

Ex. 4. If the cosines of two of the angles of a triangle are inversely proportional to the opposite sides, show that the triangle is either isosceles or right-angled.

We have, by the question,

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{\sin B}{\sin A}.$$
 [by Art. 82]

 \therefore sin $A \cos A = \sin B \cos B$, or, sin $2A = \sin 2B$,

or, $\sin 2A - \sin 2B = 0$,

or, $2\cos(A+B)\sin(A-B)=0$.

:. either $\cos (A + B) = 0$, i.e., $(A + B) = 90^{\circ}$,

i.e., the triangle is right-augled;

or,
$$\sin (A-B)=0$$
, i.e., $A-B=0$, i.e., $A=B$,

i.e., the triangle is isosceles.

Ex. 5. If the sides of a triangle are in A.P., show that $\cot \frac{A}{2}$, $\cot \frac{B}{2}$, $\cot \frac{C}{2}$ are also in A.P.

$$\cot \frac{A}{2}$$
, $\cot \frac{B}{2}$, $\cot \frac{C}{2}$ are in A.P.,

if
$$\cot \frac{B}{2} - \cot \frac{A}{2} = \cot \frac{C}{2} - \cot \frac{B}{2}$$
.

i.e., if
$$\frac{s(s-b)}{\Delta} - \frac{s(s-a)}{\Delta} = \frac{s(s-c)}{\Delta} - \frac{s(s-b)}{\Delta}$$
,
i.e., if $(s-b) - (s-a) = (s-c) - (s-b)$,
i.e., if $a-b=b-c$.

i.e., if a, b, c are in A.P.

Ex. 6. Show that

$$b^2 \sin 2C + c^2 \sin 2B = 4\Delta$$
.

Left side = $b^2.2 \sin C \cos C + c^2.2 \sin B \cos B$

$$=2b \sin C.b \cos C + 2c \sin B.c \cos B$$

$$=2b \sin C (b \cos C + c \cos B)$$

[:
$$c \sin B = b \sin C$$
]

$$=4.\frac{1}{2}ab \sin C = 4\Delta$$
. [by Art. 88]

Examples XIV(a)

In any triangle, prove that (Ex. 1 to 21):-

1.
$$\sin \frac{B-C}{2} = \frac{b-c}{a} \cos \frac{A}{2}$$
. [H. S. 1961 Comp.]

2.
$$\cos \frac{B-C}{2} = \frac{b+c}{a} \sin \frac{A}{2}$$
 [H. S. 1961 Comp.]

3.
$$(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$$
.

4.
$$\frac{a+h}{a-b} = \tan \frac{A+B}{2} \cot \frac{A-B}{2}$$

5.
$$a^2 + b^2 + c^2 = 2(bc \cos A + ca \cos B + ab \cos C)$$
.

6.
$$(b+c-a) \tan \frac{A}{2} = (c+a-b) \tan \frac{B}{2} = (a+b-c) \tan \frac{C}{2}$$

7.
$$\frac{a \sin (B-C)}{b^2-c^2} = \frac{b \sin (C-A)}{c^2-a^2} = \frac{c \sin (A-B)}{a^2-b^2}$$

8.
$$a^2 (\sin^2 B - \sin^2 C) + b^2 (\sin^2 C - \sin^2 A) + c^2 (\sin^2 A - \sin^2 B) = 0.$$

9.
$$a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) + c^2 (\cos^2 A - \cos^2 B) = 0.$$

10.
$$\frac{a^2 \sin{(B-C)}}{\sin{B} + \sin{C}} + \frac{b^2 \sin{(C-A)}}{\sin{C} + \sin{A}} + \frac{c^2 \sin{(A-B)}}{\sin{A} + \sin{B}} = 0.$$

11.
$$a \sin \frac{A}{2} \sin \frac{B-C}{2} + b \sin \frac{B}{2} \sin \frac{C-A}{2} + c \sin \frac{C}{2} \sin \frac{A-B}{2} = 0.$$

12.
$$\frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0.$$

13.
$$a^3 \sin (B-C) + b^3 \sin (C-A) + c^3 \sin (A-B) = 0$$
.

14.
$$a^3 \cos (B-C) + b^3 \cos (C-A) + c^3 \cos (A-B) = 3abc$$

15.
$$\frac{a^2 \sin (B-C)}{\sin A} + \frac{b^2 \sin (C-A)}{\sin B} + \frac{c^2 \sin (A-B)}{\sin C} = 0.$$

16.
$$(b^2-c^2)$$
 cot $A+(c^2-a^2)$ cot $B+(a^2-b^2)$ cot $C=0$.

17.
$$\frac{b^2-c^2}{\cos B+\cos C}+\frac{c^3-a^2}{\cos C+\cos A}+\frac{a^2-b^2}{\cos A+\cos B}=0.$$

18.
$$(s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$$

19.
$$\frac{b-c}{a}\cos^2\frac{A}{2} + \frac{c-a}{b}\cos^2\frac{B}{2} + \frac{a-b}{c}\cos^2\frac{C}{2} = 0$$
.

20.
$$bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2} = s^2$$
.

21.
$$\frac{1}{a}\cos^2\frac{A}{2} + \frac{1}{b}\cos^2\frac{B}{2} + \frac{1}{c}\cos^2\frac{C}{2} = \frac{s^2}{abc}$$

22. If A be 60°, show that
$$b+c=2a\cos\frac{B-C}{2}$$
.

- 23. Show that a triangle having its sides equal to 3, 5, 7 is an obtuse-angled triangle and determine the obtuse angle.
 - 24. Given (a+b+c)(b+c-a)=3bc, find A.
 - 25. If $c^4 2(a^2 + b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$, prove that $C = 60^\circ$, or, 120°.
 - 26. If $a^4 + b^4 + c^4 = 2c^2 (a^2 + b^2)$, prove that $C = 45^\circ$, or, 135°.
- 27. The sides of a triangle are 2x + 3, $x^2 + 3x + 3$, $x^2 + 2x$; show that the greatest angle is 120° .
 - 28. If $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$, show that $C = 60^\circ$.
 - 29. If a=2b and A=3B, find the angles of the triangle.
- 30. If the cosines of two of the angles of a triangle are proportional to the opposite sides, show that the triangle is isosceles.
 - 31. If $\cos A = \frac{\sin B}{2 \sin C}$ show that the triangle is isosceles.
- 32. If $(a^2 + b^2) \sin (A B) = (a^2 b^2) \sin (A + B)$, prove that the triangle is either isosceles or right-angled.
- \checkmark 33. If (cos A+2 cos C): (cos A+2 cos B) = sin B: sin C, prove that the triangle is either isosceles or right-angled.
- 34. If a^2 , b^2 , c^2 be in A.P., prove that cot A, cot B, cot C are also in A.P.
- 35. If $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$, show that the sides of the triangle are in A.P.
- 36. If $\sin A : \sin C = \sin (A B) : \sin (B C)$, show that a^2 , b^2 , c^2 are in A.P.

37. If a, b, c are in A.P., show that $\cos A \cot \frac{1}{2}A$, $\cos B \cot \frac{1}{2}B$, $\cos C \cot \frac{1}{2}C$ are in A.P. [$\cos A \cot \frac{1}{2}A = (1-2\sin^2\frac{1}{2}A)\cot\frac{1}{2}A = \cot\frac{1}{2}A - \sin A$.]

38. Assuming $\Delta = \frac{1}{2}bc \sin A$ and using the value of $\cos A$ in terms of sides, show that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

39. Find the area of the triangle whose sides are

$$\frac{y}{z} + \frac{z}{x}, \frac{z}{x} + \frac{x}{y}, \frac{x}{y} + \frac{y}{z}.$$

40. In a triangle, if a = 13, b = 14, c = 15, find its area. Prove that in any triangle:

41.
$$\frac{a^2-b^2}{2} \cdot \frac{\sin A \sin B}{\sin (A-B)} = \Delta$$
.

- 42. $4\triangle (\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$.
- 43. $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C$
- 44. $a \sin B \sin C + b \sin C \sin A + c \sin A \sin B = \frac{3\Delta}{R}$
- 45. $(a \sin A + b \sin B + c \sin C)^2$ = $(a^2 + b^2 + c^2)(\sin^2 A + \sin^2 B + \sin^2 C)$.
- 46. $\frac{\cos B \cos C}{bc} + \frac{\cos C \cos A}{ca} + \frac{\cos A \cos B}{ab} = \frac{1}{4R^3}.$

[Use Σ cot B cot C=1; ex. 2, Ex. X.]

47.
$$\frac{b^2-c^2}{a}\cos A + \frac{c^3-a^3}{b}\cos B + \frac{a^3-b^3}{c}\cos C = 0.$$

48.
$$\frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ca} = \frac{\cos C}{c} + \frac{c}{ab}.$$

49. $4\triangle = a^2 \cot A + b^2 \cot B + c^2 \cot C$.

50.
$$\left(\frac{a^2}{\sin A} + \frac{b^2}{\sin B} + \frac{c^2}{\sin C}\right) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \triangle.$$

92. Circum-radius of a triangle.

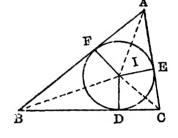
From Art. 82, we have

$$\frac{\mathbf{a}}{\sin \mathbf{A}} = \frac{\mathbf{b}}{\sin \mathbf{B}} = \frac{\mathbf{c}}{\sin \mathbf{C}} = 2\mathbf{R}. \qquad \cdots \quad (i)$$

Hence,
$$R = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A} = \frac{abc}{4\Delta}$$
 ... (ii)

93. In-radius of a triangle.

Let I be the centre and r the radius of the circle inscribed in the triangle ABC; let D, E, F be the points of contact of the in-circle with the sides BC, CA, AB respectively.



Then,
$$ID = IE = IF = r$$
.

Join IA, IB, IC.

$$\triangle ABC = \triangle IBC + \triangle ICA + \triangle IAB$$

$$= \frac{1}{2}BC.ID + \frac{1}{2}CA.IE + \frac{1}{2}AB.IF$$

$$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$= \frac{1}{2}r(a + b + c) = rs.$$

Thus, $\Delta = rs$.

Again,
$$a = BC = BD + DC$$

 $= r \cot \frac{1}{2}B + r \cot \frac{1}{2}C$, from $\triangle^s IBD$, ICD ,
 $= r \left[\frac{\cot \frac{1}{2}B}{\sin \frac{1}{2}B} + \frac{\cos \frac{1}{2}C}{\sin \frac{1}{2}C} \right]$
 $= r \left[\frac{\cos \frac{1}{2}B \sin \frac{1}{2}C + \sin \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}B \sin \frac{1}{2}C} \right]$
 $= r \frac{\sin (\frac{1}{2}B + \frac{1}{2}C)}{\sin \frac{1}{2}B \sin \frac{1}{2}C} = r \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}B \sin \frac{1}{2}C}$

$$[: : \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C = 90^{\circ}, \sin(\frac{1}{2}B + \frac{1}{2}C) = \sin(90^{\circ} - \frac{1}{2}A) = \cos(\frac{1}{2}A.]$$

 $\therefore r = a \sin \frac{1}{2}B \sin \frac{1}{2}C \sec \frac{1}{2}A = a \frac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A}.$

Since, by Art. 92(i), $a=2R \sin A=4R \sin \frac{1}{2}A \cos \frac{1}{2}A$,

 $\therefore r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C. \quad \cdots \quad (ii)$

Since, from the figure, AF = AE, BD = BF, CD = CE and since the sum of these six quantities is equal to the perimeter,

$$\therefore$$
 $AF + BD + CD = \text{semi-perimeter} = s.$

i.e.,
$$AF+BC$$
, or, $AF+a=s$.

$$\therefore AF = s - a = AE$$
.

Similarly, BF = s - b = BD; CE = s - c = CD.

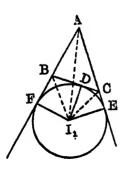
From $\triangle AIF$, $IF = AF \tan IAF$.

Similarly,
$$\mathbf{r} = (\mathbf{s} - \mathbf{a}) \tan \frac{1}{2} \mathbf{A}$$
.
Similarly, $\mathbf{r} = (\mathbf{s} - \mathbf{b}) \tan \frac{1}{2} \mathbf{B}$.
and $\mathbf{r} = (\mathbf{s} - \mathbf{c}) \tan \frac{1}{2} \mathbf{C}$.

Note. Distances of the in-centre from the vertices.

From $\triangle AIF$, IA = IF cosec IAF. ... IA = r cosec $\frac{1}{2}A$. Similarly, IB = r cosec $\frac{1}{2}B$ and IC = r cosec $\frac{1}{2}C$.

94. Ex-radii of a triangle.



Let I_1 be the centre and r_1 the radius of the escribed circle (opposite to the angle A) of the $\triangle ABC$; let D, E, F be the points of contact of the circle with the sides BC, and AC and AB produced.

Let r_3 , r_5 denote the radii of the escribed circles opposite to the angles B and C respectively.

Putting
$$a = 2R \sin A = 4R \sin \frac{1}{2}A \cos \frac{1}{2}A$$
,
 $r_1 = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

Similarly,
$$r_2 = 4R \cos \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C$$
, (ii)
and $r_3 = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C$.

Again,
$$AE = AC + CE = b + CD$$
 [: $CE = CD$]

and
$$AF = AB + BF = c + BD$$
, [: $BF = BD$]

But
$$AE = AF$$
; therefore, by addition, we get $2AE = b + c + BI + CD = b + c + a = 2s$.

$$\therefore AE = s.$$

Again, from $\triangle AI_1E$, $I_1E = AE \tan I_1AE$.

Similarly,
$$r_2 = s \tan \frac{1}{2}A$$
.
Similarly, $r_2 = s \tan \frac{1}{2}B$, and $r_3 = s \tan \frac{1}{2}C$. (iii)

Note. Distances of ex-centres from the vertices.

From $\triangle AI_1F$, $I_1A = I_1F$ cosec I_1AF .

$$I_1A = r_1 \operatorname{cosec} \frac{1}{2}A$$

$$= 4R \operatorname{cos} \frac{1}{2}B \operatorname{cos} \frac{1}{2}C. \quad \{ \text{ by formula (ii) } \}$$

From $\triangle BI$, F, I, B=I, F cosec I, BF.

:.
$$I_1B=r_1 \sec \frac{1}{2}B$$
 [:: $\angle I_1BF=90^\circ -\frac{1}{2}B$]

Similarly, $I.C=r_1 \sec \frac{1}{2}C$.

In the same way, $I_2B=r_2$ cosec $\frac{1}{2}B$, $I_3C=r_2$ cosec $\frac{1}{2}C$.

Ex. 1. Prove that
$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_5} = \frac{1}{r}$$
.

By formula (i), Art. 94,

left side =
$$\frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$$

= $\frac{3s-(a+b+c)}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$.

Ex. 2. Prove that $4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C = \frac{s}{R}$.

Left side = 4.
$$\sqrt{\frac{s(s-a)}{bc}}$$
. $\sqrt{\frac{s(s-b)}{ca}}$. $\sqrt{\frac{s(s-c)}{ab}}$
= $\frac{4s}{abc}$. $\sqrt{s(s-a)(s-b)(s-c)}$
= $\frac{4s}{abc}$. $\triangle = s$. $\frac{4\triangle}{abc} = \frac{s}{B}$ by formula (ii), Art. 92.

Ex. 3. Show that

$$bc - r_2 r_3 = ca - r_3 r_1 = ab - r_1 r_2,$$

$$r_1 \qquad r_2 \qquad r_3$$

$$r_2 r_3 = \frac{\triangle^2}{(s-b)(s-c)} = s(s-a),$$

$$bc - r_2 r_3 = \frac{1}{4} \left[4bc - 2s (2s-2a) \right]$$

$$= \frac{1}{4} \left[4bc - (a+b+c)(b+c-a) \right]$$

$$= \frac{1}{4} \left[4bc + a^2 - (b+c)^2 \right] = \frac{1}{4} \left[a^2 - (b-c)^2 \right]$$

$$= \frac{1}{4} \left[(a+b-c)(a-b+c) \right] = (s-b)(s-c).$$

$$\frac{bc - r_2 r_3}{r_1} = \frac{(s-b)(s-c)}{r_1} = \frac{\triangle}{r_1} = r.$$

Similarly, the other ratios are equal to the same quantity.

Ex. 4. Prove that in any triangle,

$$r_{1} + r_{2} + r_{3} - r = 4R.$$
Left side = $\left(\frac{\triangle}{s-a} + \frac{\triangle}{s-b}\right) + \left(\frac{\triangle}{s-c} - \frac{\triangle}{s}\right)$
= $\Delta \cdot \frac{2s - (a+b)}{(s-a)(s-b)} + \Delta \cdot \frac{c}{s(s-c)}$
= $\Delta c \left[\frac{1}{(s-a)(s-b)} + \frac{1}{s(s-c)}\right]$ [: $2s = a+b+c$.]
= $\Delta c \left[\frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$

Now, Numerator =
$$2s^2 - s(a+b+c) + ab$$

= $2s^2 - s \cdot 2s + ab = ab$.

Denominator = Δ^2 .

$$\therefore \quad \text{left side} = \frac{abc}{\Delta} = 4R.$$

Ex. 5. If $r_1 = r_2 + r_3 + r$, prove that the triangle is right-angled.

From the given relation, we have

$$r_1 - r = r_2 + r_3,$$
or,
$$\frac{\triangle}{s - a} - \frac{\triangle}{s} = \frac{\triangle}{s - b} + \frac{\triangle}{s - c},$$
or,
$$\frac{\triangle .a}{s(s - a)} = \frac{\triangle(2s - b - c)}{(s - b)(s - c)} = \frac{\triangle .a}{(s - b)(s - c)}.$$

$$\therefore s(s - a) = (s - b)(s - c).$$

$$\therefore \tan^2 \frac{1}{2}A = \frac{(s - b)(s - c)}{s(s - a)} = 1. \quad \therefore \tan \frac{1}{2}A = 1.$$

Note. Although we get tan $\frac{1}{2}A = \pm 1$, we reject the negative value because $\frac{1}{2}A$ is an acute angle.

Examples XIV(b)

Prove that in any triangle (Ex. 1 to 14):-

1.
$$\sin A + \sin B + \sin C = \frac{s}{R}$$

2.
$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

[Use $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.]

3.
$$\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$$
.

4.
$$r_2r_3+r_3r_1+r_1r_2=s^2$$
.

5.
$$r = R (\cos A + \cos B + \cos C - 1)$$
.

6.
$$r_1 = R (\cos B + \cos C - \cos A + 1)$$
.

[Use $\cos B + \cos C - \cos A = -1 + 4 \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$]

7.
$$a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \frac{\Delta}{R}$$

8.
$$a \cot A + b \cot B + c \cot C = 2(R + r)$$
.

9.
$$R = \frac{1}{4} \frac{(r_2 + r_3)(r_3 + r_1)(r_1 + r_2)}{r_2 r_3 + r_3 r_1 + r_1 r_2}.$$

10.
$$\triangle = \sqrt{rr_1r_2r_3} = r^2 \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$$

11.
$$\binom{1}{r} - \frac{1}{r_1} \binom{1}{r} - \frac{1}{r_2} \binom{1}{r} - \frac{1}{r_3} = \frac{4R}{r^2 s^2} - \frac{16R}{r^2 (a+b+c)^2}$$
[A. I. 1938]

12.
$$\left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)^2 = \frac{4}{r} \left(\frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_3}\right)$$

13.
$$r_1 (r_2 + r_3)$$
 cosec $A = r_2 (r_3 + r_1)$ cosec $B = r_3 (r_1 + r_2)$ cosec C .

14.
$$\frac{bc}{r_1} + \frac{ca}{r_2} + \frac{ab}{r_3} = 2R \left\{ \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} - 3 \right\}$$

- 15. In a triangle, a = 13, b = 14, c = 15; find r and R.
- 16. If a, b, c are in A.P., show that r_1, r_2, r_3 are in H.P.
- 17. If in a triangle, 3R = 4r, show that $4(\cos A + \cos B + \cos C) = 7$.
- 18. If the diameter of an ex-circle be equal to the perimeter of the triangle, show that the triangle is right-angled.

[Use
$$r_1 = s \tan \frac{1}{2}A$$
.]

- 19. If $\left(1 \frac{r_1}{r_2}\right) \left(1 \frac{r_1}{r_3}\right) = 2$, show that the triangle must be right-angled.
- 20. If $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right-angled.
- 21. If S be the area of the in-circle and S_1 , S_2 , S_3 are the areas of the escribed circles, then

$$\frac{1}{\sqrt{S}} = \frac{1}{\sqrt{S_1}} + \frac{1}{\sqrt{S_2}} + \frac{1}{\sqrt{S_3}}.$$

- 22. In any triangle, prove that the area of the in-circle is to the area of the triangle as π : cot $\frac{1}{2}A$ cot $\frac{1}{2}B$ cot $\frac{1}{2}C$.
- 23. If p_1 , p_2 , p_3 are the perpendiculars from the angular points of a triangle to the opposite sides, show that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

24. If x, y, z be the lengths of the perpendiculars from the circum-centre on the sides BC, CA, AB of the triangle ABC, prove that

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}.$$

25. If x, y, z are respectively equal to IA, IB, IC, and α , β , γ are respectively equal to I_1A , I_2B , I_3C , show that

)
$$\frac{xyz}{abc} = \frac{r}{s}$$
 (ii) $\frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$.

(iii)
$$\frac{bc}{a^{\frac{1}{2}}} + \frac{ca}{\beta^{\frac{2}{3}}} + \frac{ab}{\gamma^{\frac{2}{3}}} = 1$$
. (iv) $ax^{2} + by^{2} + cz^{2} = abc$.

[Use Notes of Arts. 93 and 94.]

26. Prove that

$$(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$$

- 27. If $\frac{r}{2} = \frac{r_1}{12} = \frac{R}{\Delta}$ prove that the Δ is right angled.
- 28. If I be the in-centre and x, y, z be the circumradii of the triangles BIC, CIA, AIB, then show that $xyz = 2R^2r$.

29. If I be the in-centre of the triangle ABC, show that

$$IA.IB.IC = \frac{abc}{s} \triangle$$

where Δ denotes the area of ΔABC and s is its semi-perimeter.

30. If D, E, F be the points of contact of the in-circle with the sides BC, CA, AB respectively, show that

$$EF = 2(s - a) \sin \frac{1}{2}A$$
,
 $FD = 2(s - b) \sin \frac{1}{2}B$,
 $DE = 2(s - c) \sin \frac{1}{2}C$.

- 31. If the lengths of the internal bisectors of the angle of a triangle be equal, show that the triangle is isosceles.
- 32. If x, y, z be the lengths of the internal bisetors of the angles of a triangle and l, m, n be the lengths of those bisectors, produced to meet the circle, show that

(i)
$$\frac{\cos \frac{1}{2}A}{x} + \frac{\cos \frac{1}{2}B}{y} + \frac{\cos \frac{1}{2}C}{z} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

- (ii) $l \cos \frac{1}{2}A + m \cos \frac{1}{2}B + n \cos \frac{1}{2}C = a + b + c$.
- 33. If the internal bisectors of the angles of a triangle make angles α , β and γ with the sides BC, CA and AB respectively, show that

$$a \sin 2a + b \sin 2\beta + c \sin 2\gamma = 0$$
.

34. If l, m, n be the lengths of the medians of a triangle ABC, prove that

(i)
$$(b^2-c^2)l^2+(c^2-a^2)m^2+(a^2-b^2)n^2=0$$
.

(ii)
$$4(l^2 + m^2 + n^2) = 3(a^2 + b^2 + c^2)$$
.

- 35. Perpendiculars AL, BM, CN are drawn from the vertices A, B, C of an acute-angled triangle on the opposite sides and produced to meet the circumcircle of $\triangle ABC$ in L', M', N'. If LL', MM', NN' be equal to a, β , γ respectively, show that
 - (i) $\frac{a}{a} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$.

(ii)
$$\frac{AL'}{AL} + \frac{BM'}{BM} + \frac{CN'}{CN} = 4$$
.

- 36. If x, y, z are respectively equal to IA, IB, IC and x_1 , y_1 , z_1 are respectively equal to I_1A , I_2B , I_3C where I is the in-centre and I_1 , I_2 , I_3 are then ex-centres of $\triangle ABC$, then show that
 - (i) $\frac{b-c}{ax_1^2} + \frac{c-a}{by_1^2} + \frac{a-b}{cz_1^2} = 0$.

(ii)
$$x^2 \left(\frac{1}{b} - \frac{1}{c} \right) + y^2 \left(\frac{1}{c} - \frac{1}{a} \right) + z^2 \left(\frac{1}{a} - \frac{1}{b} \right) = 0.$$

37. Prove that in the triangle ABC

$$\frac{IA}{I_1A} + \frac{IB}{I_2B} + \frac{IC}{I_3C} = 1.$$

38. Prove that in the triangle ABC

$$a.BP.CP + b.CP.AP + c.AP.BP = abc$$

where P is the orthocentre of the triangle

39. If AL, BM, CN are the perpendiculars from the angular points of $\triangle ABC$ to the opposite sides, show that

$$\frac{1}{AL} + \frac{1}{BM} + \frac{1}{CN} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

40. In the triangle ABC (with the usual notation), show that $\frac{r_1r_2r_3}{r^3} = \frac{(a+b+c)^3}{(b+c-a)(c+a-b)(a+b-c)}.$

CHAPTER XV

SOLUTION OF TRIANGLES

- 95. In a triangle, there are six parts, the three sides and the three angles. These are not independent, but are connected by the relations between the sides and angles of the triangle, which have been established in Chapter XIV. In fact, if three of the parts are given, the remaining three can, in general, be determined, and the corresponding triangle completely known. The cases that can arise are the following:
 - (1) three sides may be given;
 - (2) three angles may be given;
 - (3) two sides and the included angle may be given;
 - (4) two angles and one side may be given;
 - (5) two sides and an opposite angle may be given.

We shall discuss these cases one by one.

96. Three sides given.

Let the three sides a, b, c of a triangle ABC be given. Now, provided the sum of any two of these given sides is greater than the third, the triangle ABC with the three given sides can be geometrically constructed and the triangle is unique; in other words, its angles are definite. To determine any angle, say A, we may use the rigorous formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and thereby determine $\cos A$, and then from the cosine-table find out the angle with this cosine. It is clear that the angle, being an angle of a triangle, lies between 0 and π , and within this range an angle with a given cosine has got only one value. Thus the angle is definitely known.

Here we want to make one point clear. Though the formula used is rigorous, the cosine-table, by means of which we determine the angle with a given cosine, gives only approximate values. Now, it is a principle proved in books on higher mathematics (with the aid of Calculus), that when an angle is determined by using an approximate table the best result is obtained by using the Logarithmic tangent-table, and an angle determined from its L tan, using a four-figure table is more accurate than that determined by using even a seven-figure sine-table or cosine-table. If a suitable tangent formula is available, therefore, we should make use of it.

Hence, for practical purposes, in this case, to determine A, we use the formula,

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

where $s = \frac{1}{2}(a + b + c)$, which is known.

Taking logarithm, and adding 10, we get the value of L tan $\frac{1}{2}A$ and therefore A is known.

Similarly, B and C are determined.

In case $\tan \frac{1}{2}A$ happens to be equal to the tangent of a standard angle, $\frac{1}{2}A$ is at once known and the use of logarithm is not wanted.

Ex. The sides of a triangle are 2, 3, 4. Find the greatest angle, having given

$$log \ 2 = 30103, log \ 3 = 4771213,$$

 $L \ tan \ 52^{\circ} \ 14' = 10'1108395, L \ tan \ 52^{\circ} \ 15' = 10'1111004.$

Here,
$$s = \frac{2+3+4}{2} = \frac{9}{2}$$
.

The greatest side 4 being denoted by 'a', the greatest angle A (which is opposite to the greatest side) is given by

$$\tan \frac{1}{2}A = \sqrt{\frac{(\frac{9}{2} - 2)(\frac{9}{2} - 3)}{\frac{9}{2}(\frac{9}{2} - 4)}} = \sqrt{\frac{5.3}{9.1}} = \sqrt{\frac{10}{2.3}}.$$

$$\therefore L \tan \frac{1}{2}A = 10 + \frac{1}{2} (\log 10 - \log 2 - \log 3)$$
$$= 10 + \frac{1}{2} (1 - 30103 - 4771213)$$
$$= 10.1109244.$$

Now, $L \tan \frac{1}{2}A$ lies between $L \tan 52^{\circ} 14'$ and $L \tan 52^{\circ} 15'$.

Hence, $\frac{1}{2}A$ lies between 52° 14′ and 52° 15′.

Let $\frac{1}{2}A = 52^{\circ} 14' x''$.

Then diff. for a'' is '0000849,

and diff. for 1' i.e., 60" is '0002609.

Hence,
$$\frac{x}{60} = \frac{849}{2609}$$
, or, $x = \frac{60 \times 849}{2609} = 195$ nearly.

Hence, $\frac{1}{2}A = 52^{\circ} 14' 19'' 5$,

or,
$$A = 104^{\circ} 28' 39''$$
 nearly.

97. Three angles given.

In this case the triangle cannot be solved, for there are innumerable triangles with the same three angles. All these triangles, being equiangular, are similar, and the ratio of their sides can be determined from the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or. $a:b:c=\sin A:\sin B:\sin C$.

Ex. The angles of a triangle are in the ratio 2:3:7. Prove that the sides are in the ratio $\sqrt{2}:2:(\sqrt{3}+1)$.

The angles being in the ratio 2:3:7, and their sum being 180°, the angles are evidently 30°, 45° and 105° respectively. Hence, the ratio of the sides will be

sin 30°: sin 45°: sin 105°.

i.e.,
$$\frac{1}{2}: \frac{1}{\sqrt{2}}: \frac{\sqrt{3+1}}{2\sqrt{2}}$$
,

or, $\sqrt{2}:2:(\sqrt{3}+1)$.

Examples XV(a)

- 1. The sides of a triangle are 24, 22, 14; find the least angle, given $L \tan 17^{\circ} 33' = 9.500042$, diff. for 1' = 439.
- 2. The sides of a triangle are 50, 36 and 28; find the greatest angle, having given

$$\log 10 = 1.2787536$$
, $\log 29 = 1.4623980$

 $L \tan 51^{\circ}0' = 10^{\circ}0916308$, $L \tan 50^{\circ}1' = 10^{\circ}0918891$.

3. The sides of a triangle are 9, 10 and 11; find the angle opposite to the side 10, given

L tan 29° 30′ = 9'7526420, L tan 29° 29′ = 9'7523472,
$$\log 2 = 30103$$
. [C. U. 1943]

4. The sides of a triangle are 2, 3, 4. Find all the angles correct to degrees and minutes by the help of mathematical tables.

5. (i) The sides of a triangle are 15, 19, 24; find the greatest angle of the triangle.

Given log 5.7 = .75587, $L \cos 88^{\circ} 59' = 8.24903$ diff. for 1' = .718. [C. U. 1936]

(ii) Find the greatest angle in degrees, minutes and seconds in a triangle whose sides are 5, 6, 7, having given $\cos 6 = 7781513$

 $L \cos 39^{\circ} 14' = 9.8890644$, diff. for 60'' = .0001032.

- 6. (i) The sides of a triangle are 7, 8, 9; solve the triangle. [C. U. 1938]
- (ii) If a = 35, b = 40, c = 66, determine the greatest angle. [C. U. 1945]

[Use Mathematical Tables]

- 7. Given $a = \sqrt{6}$, b = 2, $c = \sqrt{3} 1$; solve the triangle.
- 8. Given a=2, $b=\sqrt{2}$, $c=\sqrt{3}+1$; solve the triangle.
- 9. If a = 7, b = 5, c = 8, solve the triangle. Given $\cos 38^{\circ} 11' = \frac{1}{12}$.
- 10. If $a = 3 + \sqrt{3}$, $b = 2\sqrt{3}$, $c = \sqrt{3}$, solve the triangle.
- 11. The angles of a triangle are 105°, 60° and 15°; find the ratio of the sides.
 - 12. If $A = 45^{\circ}$, $B = 60^{\circ}$, show that $c : a = \sqrt{3+1} : 2$.
- 13. The angles of a triangle are as 1:2:7; find the ratio of the greatest side to the least side.
 - 14. If $\cos A = \frac{4}{8}$, $\cos B = \frac{3}{8}$, find a : b : c.
- 15. If the angles adjacent to the base of a triangle are $22\frac{1}{2}^{\circ}$ and $112\frac{1}{2}^{\circ}$, show that the altitude is half the base.
- 16. If the sides of a triangle are 4, 5, 6, show that the greatest angle is double the least.

98. Two sides and the included angle given.

Let the two sides b, c and the included angle A of a triangle ABC be given. It is easy to construct the triangle geometrically, and there will be only one definite triangle with the given parts. To find the other angles B and C, we notice that

$$B+C=180^{\circ}-A,$$

i.e., $\frac{B+C}{2}=90^{\circ}-\frac{A}{2}.$

Again,

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

$$\therefore L \tan \frac{B-C}{C} = 10 + \log \left(\frac{b-c}{b+a} \cot \frac{A}{2}\right)$$

$$= \log \left(\frac{b-c}{b+c}\right) + L \cot \frac{A}{2}.$$

b, c, and A being given, the right-hand side is known and thus, $L \tan \frac{B-C}{2}$ is known, whence $\frac{B-C}{2}$ is known.

Now $\frac{B+C}{2}$ and $\frac{B-C}{2}$ being both known, by addition and subtraction, we get B and C respectively.

The reason of using tangent formula to determine $\frac{B-C}{2}$ is already explained in Art. 96.

When B and C are known, the third side a is easily obtained from

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or, } = \frac{c}{\sin C}$$

Ex. In a triangle, $h=2^{\circ}25$, $c=1^{\circ}75$, $\Lambda=54^{\circ}$, find B and C, having given,

Here.

$$\frac{B+C}{2} = 90^{\circ} - \frac{A}{2} = 90^{\circ} - 27^{\circ} = 63^{\circ}.$$
 (i)

Again,

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{5}{4} \cot 27$$
$$= \frac{1}{8} \tan 63^{\circ}.$$

$$L \tan \frac{B-C}{2} = L \tan 63^{\circ} - 3 \log 2$$

$$= 10^{\circ}292834 - 903090$$

$$= 9^{\circ}389744.$$

Now, $L \tan 13^{\circ} 47' = 9.389724$ and $L \tan 13^{\circ} 48' = 9.390270$.

we get, diff. for x'' = 000020 and diff. for 1' i.e., 60'' = 000546,

$$\therefore \frac{x}{60} = \frac{20}{546}$$
, or, $x = \frac{20 \times 60}{546} = 2.2$ nearly.

Hence, $\frac{B-C}{2} = 13^{\circ} 47' 2'''2$ nearly.

Combining with (i), $B = 76^{\circ} 47' 2''' 2$ and $C = 49^{\circ} 12' 57''' 8$.

99. Two angles and a side given.

Let any side a of a triangle ABC, and any two of its angles be given. The sum of the three angles being 180°, the third angle is also known. To find the other two sides b and c, we use the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Ex. In a triangle ABC, $A = 38^{\circ} 20'$, $B = 45^{\circ}$ and b = 64 ft. Find c, having given log 2 = 30103, L sin $83^{\circ} 20' = 999705$ and log $089896 = \overline{2}95374$.

Here.

$$C = 180^{\circ} - (A + B) = 180^{\circ} - 83^{\circ} 20'$$
.

Now.

$$\frac{c}{\sin C} = \frac{b}{\sin B},$$

or,
$$\frac{c}{\sin{(180^{\circ}-83^{\circ}20')}} = \frac{64}{\sin{45^{\circ}}} = \frac{64}{1/\sqrt{!}} \cdot 64\sqrt{2}$$
.

$$c = 2^{\frac{18}{2}} \sin 83^{\circ} 20'.$$

Thus, $\log c$ has the same mantissa as $\log 089896$, but has its characteristic 1. Hence, c = 89896 feet.

Examples XV(b)

1. Two sides of a triangle are 3 and 5 feet and the included angle is 120°; find the other angles, having given log 4'8 = '6812412

$$L \tan 8^{\circ} 12' = 9'1586706$$
, diff. for $60'' = 8940$.

[C. U. 1949]

- 2. If b = 1300, c = 1400 and $A = 60^{\circ}$, find B and C. Given $\log 3 = 4771213$, L tan $3^{\circ} 40' = 88067422$, diff. for 10'' = 3306.
- 3. If a=21, b=11, $C=34^{\circ} 42' 30''$, find A and B. Given $\log 2=30103$, and L tan $72^{\circ} 38' 45'' = 10.50515$.
- 4. If the sides a and b are in the ratio 7: 3 and the included angle C is 60°, find A and B, given

$$\log 2 = 3010300, \qquad \log 3 = 4771213$$

 $L \tan 34^{\circ} 42' = 9.8403776$, diff. for 1' = 2699.

5. Two sides of a plane triangle are 14 and 11 and the included angle is 60° . Find the remaining angles, having given L tan 11° 44' = 9.3174299, L tan 11° 45' = 9.3180640.

[C. U. 1922]

- 6. (i) Two sides of a triangle are 80 and 100 ft. and the included angle is 60°. Find the other angles. [C. U. 1946]
 - (ii) If a = 5, b = 3, $C = 70^{\circ}$ 30', find the remaining angles.
 - (iii) If a = 39.9, b = 43.2, $C = 38^{\circ} 14'$, solve the triangle.

 [Use Mathematical Tables]
- 7. (i) In a plane triangle, b = 540, c = 420 and $A = 52^{\circ}$ 6'; find B and C, having given

 $L \tan 26^{\circ} 3' = 9.6891430.$

 $L \tan 14^{\circ} 20' = 9.4074189$.

 $L \tan 14^{\circ} 21' = 9.4079453.$ [C. U. 1934]

- (ii) Given a = 70, b = 35, $C = 36^{\circ} 52' 12''$, $\log 3 = 0.4771213$, $L \cot 18^{\circ} 26' 6'' = 10.4771213$. Calculate the other two angles A and B. [C. U. 1935, '37]
 - 8. If $a=2\sqrt{6}$, $c=6-2\sqrt{3}$, $B=75^{\circ}$, solve the triangle.
- 9. Two sides of a triangle are $\sqrt{3}+1$ and $\sqrt{3}-1$ and the included angle is 60° ; solve the triangle.
 - 10. (i) If a = 2, $b = 1 + \sqrt{3}$, $C = 60^{\circ}$, solve the triangle.
 - (ii) If a = 2, b = 4, $C = 60^{\circ}$, find A and B.
- 11. If a=19, $B=52^{\circ}$ 28' and $C=93^{\circ}$ 40', find b, having given $\log 27038=4'4319746$; $\log 19=1'2787536$; $\log 27037=4'4319585$;

 $L \sin 52^{\circ} 28' = 9.8992727$, $L \sin 33^{\circ} 52' = 9.7460595$.

12. If $B=45^{\circ}$, $C=10^{\circ}$ and a=200 ft., find b, having given $\log 2=30103$, $L \sin 55^{\circ}=99133645$, $\log 1726^{\circ}4=32371414$, $\log 1726^{\circ}5=32371666$.

[C. U. 1947]

13. If $A = 41^{\circ} 13' 22''$, $B = 71^{\circ} 19' 5''$, and a = 55, find b, given $\log 55 = 1.7403627$, $\log 79063 = 4.8979775$, $L \sin 41^{\circ} 13' 22'' = 9.8188779$

 $L \sin 41^{\circ} 13^{\circ} 22^{\circ} = 9.8188779,$ $L \sin 71^{\circ} 19^{\prime} 5^{\prime\prime} = 9.9764927.$

- 14. (i) If $B = 70^{\circ} 30'$, $C = 78^{\circ} 10'$, a = 102, solve the triangle.
 - (ii) If a = 39, $A = 81^{\circ} 35'$, $B = 27^{\circ} 55'$, solve the triangle.
 - (iii) If $A = 37^{\circ} 15'$, $B = 72^{\circ} 5'$, $a = 75^{\circ} 2$, find b and c.

 [Mathematical tables should be used]
 - 15. If $A = 75^{\circ}$, $B = 30^{\circ}$, $b = \sqrt{8}$, solve the triangle.
 - 16. If $A = 30^{\circ}$, $B = 45^{\circ}$, b = 2, solve the triangle.
- 17. In a triangle in which each base angle is double of the third angle, the base is 2; solve the triangle.
 - 18. Given $a = \sqrt{57}$, $A = 60^{\circ}$, $\Delta = 2\sqrt{3}$, find b and c.
 - 100. Two sides and an opposite angle given.

Let the two sides b and c in a triangle ABC, and the angle B opposite to the side b be given.

In this case, we get the angle C from the formula.

$$\frac{\sin C}{c} = \frac{\sin B}{b}, \text{ or, } \sin C = \frac{c \sin B}{b}.$$

Now, three cases may arise, namely.

- (i) $c \sin B > b$. In this case sin C is greater than 1, and so C cannot be obtained. In fact in this case no triangle is possible.
- (ii) $c \sin B = b$. Here, $\sin C$ becomes 1 and therefore, $C = 90^{\circ}$. Thus, $A = 90^{\circ} B$. We thus get a right-angled triangle with right angle at C, and the side a will be obtained from

$$c^2 = a^2 + b^2$$
, or, $a = \sqrt{c^2 - b^2}$.

(iii) $c \sin B < b$. In this case $\sin C$ is less than 1, and hence, C can be determined. Now, sines of supplementary angles are known to be equal, and an angle of a triangle may be acute or obtuse. We therefore get two supplementary values of C having the same value for $\sin C$. Three subcases now arise:

Sub-case A. If of the two given sides, b > c, then B > C, and therefore the obtuse value of C becomes inadmissible, for otherwise B is also obtuse and two angles B and C of the triangle become both obtuse. Thus, the only admissible solution is the acute value of C. Now, B and C being both known, A is obtained from $A + B + C = 180^{\circ}$. The side a will be known from

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
, or, $\frac{c}{\sin C}$

Thus, the triangle is uniquely solved.

Sub-case B. If b=c, then B=C, and here also the obtuse value of C is inadmissible; with the acute value of C the triangle is uniquely solved exactly as in the above case.

Sub-case C. If b < c, then B < C, so that C may be either acute or obtuse. Both the supplementary values of C being admissible now, there will be two possible triangles with the three given parts b, c, B. Corresponding to each value of C, the value of C is determined from C and then C is obtained from the formula.

As there are two solutions of the triangle in this case, each equally admissible, this sub-case in the solution of a triangle in which b, c, B are given and $b > c \sin \rho$ and c, is referred to as the Ambiguous Case in the solution of triangles.

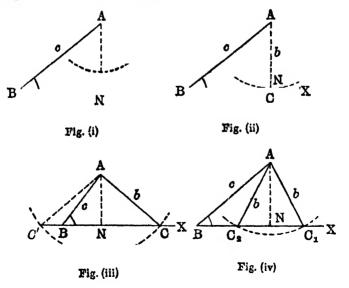
We may sum up the results as follows:

When in a triangle, b, c, B are given,

- (i) if $b < c \sin B$, no triangle is possible;
- (ii) if $b = c \sin B$, we get a definite right-angled triangle as solution;
- (iii) if b > c and therefore necessarily $> c \sin B$, we get one definite solution having C acute;
- (iv) if b=c and therefore necessarily $> c \sin B$, we get one definite solution having C acute;
- (v) if $b > c \sin B$ but < c, there are two solutions, and this case is the ambiguous case,

101. Geometrical treatment of the Ambiguous Case.

To make the ideas clear, we proceed to construct geometrically the triangle in which two sides and an opposite angle, viz., b, c and B are given.



Let ABX be the given angle B, and along one arm of it, take AB=c. Let AN be the perpendicular from A on BX. Then $\frac{AN}{AB}=\sin B$, so that $AN=AB\sin B=c\sin B$.

With centre A and radius b draw a circle.

Case (i). If $b < c \sin B$, i.e., < AN, the circle does not meet the side BX at all and no triangle is therefore obtained. [See f(g.(i))]

Case (ii). If $b = c \sin B$, i.e., = AN, the circle touches the side BX at C coincident with N, as in fig. (ii). Hence, a right-angled triangle is formed, in which the sides AB, AC and the angle B have the given values c, b, B. Thus, ABC is the required triangle.

Case (iii). If b > c, i.e., > AB, the circle cuts BX at two points C and C' on opposite sides of B as in fig. (iii). The triangle ABC', though it has the sides AB, AC equal to the given quantities c and b, has the angle B not equal to the given angle, but equal to its supplement. Hence it is not the solution required. In this case the triangle ABC is the only solution.

Case (iv). If b=c, i.e., =AB, the point C' of the above case coincides with B, and only one triangle ABC is obtained as the required solution.

Case (v). If $b > c \sin B$, i.e., > AN but less than c (or, AB), the circle cuts BX at two points C_1 and C_2 on the same side of B as in fig. (iv). Both the triangles ABC_1 and ABC_2 have the same three given parts and both are possible solutions. This is therefore the Ambiguous case.

Note. By considering the equation

$$b^2 = c^2 + a^2 - 2ac \cos B$$

in which b, c, B are given, we may first of all determine a, instead of trying to determine C.

Considering the equation as a quadratic in a, vis.,

$$a^2 - 2c \cos B \cdot a + c^2 - b^2 = 0$$

and by solving it, we tet

$$a = c \cos B + \sqrt{b^2 - c^2 \sin^2 B}$$
.

- (i) If $b < c \sin B$, $b^2 c^2 \sin^2 B$ is negative and thus the two values of a are imaginary. (No solution)
- (ii) If $b=c\sin B$, $b^2-c^2\sin^2 B=0$ and thus the two values of a are real and coincident.

(one solution: one triangle right-angled at C, since $b=c\sin B$)

- (iii) If $b > c \sin B$, $b^2 c^2 \sin^2 B$ is positive, so two values of a are real and distinct, but they are not always admissible.
- (a) When b > c, {i.e., $b^2 > c^2 (\sin^2 B + \cos^2 B)$ }, $b^2 c^2 \sin^2 B > c^2 \cos^2 B$, i.e., $\sqrt{b^2 c^2 \sin^2 B} > c \cos^2 B$ and hence one value of a is positive and the other negative. (one solution)
- (b) When b=c, $b^2-c^2\sin^2 B=c^2-c^2\sin^2 B=c^3\cos^3 B$ and hence one value of a is zero. (one solution)
- (c) When b < c, i-e., $b^2 < c^2 (\sin^2 B + \cos^2 B)$, $b^2 c^2 \sin^2 B < c^2 \cos^2 B$, i.e., $\sqrt{b^2 c^2 \sin^2 B} < c \cos B$.

So both values of a are real and positive. (two solutions)

This is known as the algebraical discussion of the ambiguous case.

An example illustrating the algebraic method is added below.

Ex. 1. In a triangle, b = 15 ft., c = 10 ft., $B = 60^{\circ}$. Find a and A having given $\sin 84^{\circ} 44' = 99578$.

We have
$$b^2 = c^2 + a^2 - 2ca \cos B$$
, giving here $225 = 100 + a^2 - 20a \cos 60^\circ$:

or,
$$a^2 - 10a - 125 = 0$$
 whence $a = 5 + 5 .6$.

Rejecting the negative value for a as inadmissible, the only possible value of a=5 ($\sqrt{6}+1$) ft. There is thus one solution and there is no ambiguity. In fact this is case (iii) of the previous article.

Again,
$$\sin A = \frac{a}{b} \sin B = \frac{5(\sqrt{6+1})}{15} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{2} + \sqrt{3}}{6}$$
$$= \frac{3 \times 1.41421 \dots + 1.73205}{6} = .99578 \dots$$

so $A = 84^{\circ} 44'$.

Ex. 2. In a triangle, $a = 73^{\circ}4$, $b = 64^{\circ}9$ and $B = 48^{\circ}13'25''$; find A, having given

log
$$734 = 2.8656961$$
, log $649 = 2.8122447$
L sin 48° $13'$ $25'' = 9.8725936$
L sin 57° $30' = 9.9260292$ (diff. for $1' = 804$)

Is the case ambiguous?

Here,

$$\sin A = \frac{a \sin B}{b} = \frac{734}{649} \sin 48^{\circ} 13' 25''.$$

$$\therefore L \sin A = \log 734 - \log 649 + L \sin 48^{\circ} 13' 25''$$

$$= 2'8656961 - 2'8122447 + 9'8725936$$

$$= 9'9260450.$$

Now, diff. of this from $L \sin 57^{\circ} 30' = 158$ (i.e., '0000158) and diff. for 1' (or 60'') = 804 (i.e., '0000804).

Hence,
$$A = 57^{\circ} 30' x''$$
 where $\frac{x}{60} = \frac{158}{804}$ whence $x = 11.8$ nearly.

Thus, $A = 57^{\circ} 30' 11'8''$ or its supplement $122^{\circ} 29' 48'2''$ which has also the same sine, and so the same L sine.

Now, in this case a > b and so A > B and thus both values of A are admissible. The case is, therefore, the ambiguous case and will have two solutions.

Examples XV(c)

1. Given (i) $A = 30^{\circ}$, a = 6, b = 4.

(ii)
$$A = 60^{\circ}$$
, $a = 7$, $b = 8$.

(iii)
$$A = 45^{\circ}$$
, $a = 2$, $b = 8$.

(iv)
$$A = 30^{\circ}$$
, $a = 3$, $b = 6$.

Find in which case the solution is ambiguous, in which case there is one solution, and in which case there is no solution.

- 2. (i) If b = 2, $c = \sqrt{3+1}$ and $B = 45^{\circ}$, solve the triangle.
 - (ii) If a = 3, $b = 3\sqrt{3}$, $A = 30^{\circ}$, find B.
- 3. If a=2, $b=\sqrt{6}$, $B=60^{\circ}$, solve the triangle.
- 4. If a=2, b=5, $A=30^{\circ}$, solve the triangle.
- 5. If b, c, B are given and if b < c, show that $(a_1 a_2)^2 + (a_1 + a_2)^2 \tan^2 B = 4b^2$,

 a_1 and a_2 being the two possible values of a.

6. In the ambiguous case, given a, b and A, prove that the difference between the two values of c is

$$2\sqrt{a^2-b^2\sin^2 A}$$
.

7. If a, b, A are given, and if c_1 , c_2 are the values of the third side in the ambiguous case, prove that if $c_1 > c_2$,

(i)
$$c_1 - c_2 = 2a \cos B_1$$
. [B. II. U. I. 1928]

(ii)
$$c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2 \cos^2 A$$
.

[B. H. U. I. 1935; Pat. I. 1936]

(iii)
$$\cos \frac{C_1 - C_2}{2} = \frac{b \sin A}{a}$$
. [A. I. 1941]

8. If b=16, c=25 and $B=33^{\circ}15'$, find the other angles; given

$$L \sin 33^{\circ} 15' = 9.7390129$$
, $\log 2 = .30103$,

 $L \sin 58^{\circ} 57' = 9.9328376$, $L \sin 58^{\circ} 56' = 9.9327616$.

- 9. If a=5, b=4, $A=45^{\circ}$, find B and C; given $\log 2=30103$, $L \sin 34^{\circ} 27'=9.75257$.
- 10. If a=30, b=300, find A in order that B may be a right angle, having given that

 $L \sin 5^{\circ} 44' = 8.9995595$, diff. for 1' = 12565.

11. If a=16, c=25 and $C=60^{\circ}$, find the other angles; given

 $\log 2 = 30103$, $\log 3 = 4771213$

 $L \sin 33^{\circ} 39' = 9.7436024$, diff. for 1' = 1897.

12. If b = 165, c = 258, and $B = 35^{\circ} 10'$, find the angles A and C; given

 $\log 1.65 = 21749$, $\log 2.58 = 41162$ $L \sin 35^{\circ} 10' = 9.76039$, $L \sin 64^{\circ} 14' = 9.95452$.

- 13. If 2b = 3a and $\tan^2 A = \frac{3}{6}$, prove that there are two values of the third side, one of which is double the other.
- 14. If A_1 , B_1 and A_2 , B_2 are the angles of the two triangles in the ambiguous case where h, c, C are given,

then $\frac{\sin A_1}{\sin B_1} + \frac{\sin A_2}{\sin B_2} = 2 \cos C$.

15. Show that in the case that admits of two solutions, the two values of C satisfy the equation

$$\frac{(a+b)^2}{1+\cos C} + \frac{(b-a)^2}{1-\cos C} = \frac{2a^2}{\sin^2 A} \cdot [B. H. U. I. 1942]$$

16. If $\log b + 10 = \log c + L \sin B$, can the triangle be ambiguous?

Miscellaneous Examples II

In any triangle ABC, prove that (Ex. 1 to 8):—

1.
$$\frac{1}{a}\cos A + \frac{1}{b}\cos B + \frac{1}{c}\cos C = \frac{a^2 + b^2 + c^2}{2abc}$$

2.
$$(b^2 + c^2 - a^2) \tan A = (c^2 + a^2 - b^2) \tan B$$

= $(a^2 + b^2 - c^2) \tan C$.

3.
$$b^2 + c^2 - 2bc \cos(A + 60^\circ) = c^2 + a^2 - 2ca \cos(B + 60^\circ)$$

= $a^2 + b^2 - 2ab \cos(C + 60^\circ)$.

4.
$$(\cot \frac{1}{2}A - \tan \frac{1}{2}B - \tan \frac{1}{2}C)^{\frac{1}{2}}$$

$$+\left(\cot \frac{1}{2}B - \tan \frac{1}{2}C - \tan \frac{1}{2}A\right)^{\frac{1}{2}} + \left(\cot \frac{1}{2}C - \tan \frac{1}{2}A - \tan \frac{1}{2}B\right)^{\frac{1}{2}}$$

$$= \left(\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C\right)^{\frac{1}{2}}.$$

5.
$$a \sin (B-C) \cos (B+C-A) + b \sin (C-A)$$

 $\times \cos (C+A-B) + c \sin (A-B) \cos (A+B-C) = 0$.

6.
$$\frac{a \sin A + b \sin B + c \sin C}{a \cos A + b \cos B + c \cos C} = \frac{R}{abc} (a^2 + b^2 + c^2).$$

7.
$$(b+c-2a) \sin \frac{1}{2}A \sin \frac{1}{2} (B-C)$$

 $+(c+a-2b) \sin \frac{1}{2}B \sin \frac{1}{2} (C-A)$
 $+(a+b-2c) \sin \frac{1}{2}C \sin \frac{1}{2} (A-B) = 0.$

- 8. $a \cos A \cos 2A + b \cos B \cos 2B + c \cos C \cos 2C + 4 \cos A \cos B \cos C (a \cos A + b \cos B + c \cos C) = 0$.
- 9. If in a triangle, a^2 , b^2 , c^2 are in A.P., show that $\tan A$, $\tan B$, $\tan C$ are in H.P.
- 10. If in a triangle, $\sin A$, $\sin B$, $\sin C$ are in H.P., show that $1 \cos A$, $1 \cos B$, $1 \cos C$ are in H.P.
- 11. Determine the triangle whose sides are three consecutive terms in the series of natural numbers and whose largest angle is double the least.

- 12. If in a triangle, $\cos 3A + \cos 3B + \cos 3C = 1$, show that one angle must be 120°.
- 13. If in a triangle, $\sin A$, $\sin B$, $\sin C$ be in A.P., show that $\tan \frac{1}{2}A \tan \frac{1}{2}C = \frac{1}{2}$.
- 14. If a=5, b=7 and $A=30^{\circ}$, find B in degrees and minutes, having given $\sin 44^{\circ} = 0.6947$, $\sin 45^{\circ} = 0.7071$.
- 15. In the ambiguous case, the area of one of the triangles is n times that of the other; show that if b be the greater of the given sides and c the less, $\frac{b}{c}$ is less than $\frac{n+1}{n-1}$.
- 16. In the ambiguous case, show that the circum-circles of the two triangles are equal.
 - 17. Prove that

(i)
$$\tan^{-1}\left(\frac{x\cos\phi}{1-x\sin\phi}\right) - \tan^{-1}\left(\frac{x-\sin\phi}{\cos\phi}\right) = \phi$$
.

(ii)
$$\tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1} \frac{t_2 - t_3}{1 + t_2 t_3} + \dots + \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n} = \tan^{-1} t_1 - \tan^{-1} t_n.$$

- 18. If the sum of four angles be 180°, prove that the sum of the products of their cosines taken two and two together is equal to the sum of the products of their sines taken similarly.
 - 19. Prove that $\cos^2 A + \cos^2 \left(A + \frac{\pi}{3}\right) + \cos^2 \left(A \frac{\pi}{3}\right) = \frac{3}{2}$.
- 20. In a triangle ABC, if $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$ be in Arithmetical Progression then $\cos A$, $\cos B$, $\cos C$ are also in Arithmetical Progression.

21. Give in general terms the solutions of the following equation:

$$\tan (x+b) \tan (x+c) + \tan (x+c) \tan (x+a)$$
$$+ \tan (x+a) \tan (x+b) = 1.$$

22. If $A + B + C = 180^{\circ}$, prove that

$$\left(1 + \tan\frac{A}{4}\right)\left(1 + \tan\frac{B}{4}\right)\left(1 + \tan\frac{C}{4}\right)$$
$$= 2\left(1 + \tan\frac{A}{4}\tan\frac{B}{4}\tan\frac{C}{4}\right).$$

23. Prove that

$$\sin^2 x + \sin^2 y + \sin^2 z + \sin^2 (x + y + z)$$

= 2 - 2 \cos (x + y) \cos (y + z) \cos (z + x).

24. Solve the following equation:

$$\tan x + \tan \left(x + \frac{\pi}{3}\right) + \tan \left(x + \frac{2\pi}{3}\right) = 3.$$

[Left side reduces to 3 tan 3x.]

25. Prove that in a triangle ABC,

$$\Delta = \frac{(a+b+c)^2}{4\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)}$$

26. Prove that

$$\log \sin 8x = 3 \log 2 + \log \sin x + \log \cos x + \log \cos 2x + \log \cos 4x.$$

27. Show that in any triangle ABC,

$$\log \tan \frac{A}{2} = \frac{1}{2} [\log (s-b) + \log (s-c) - \log s - \log (s-a)].$$

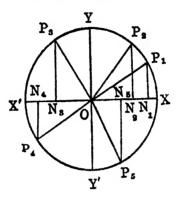
28. Prove that (i) $x^{\log y} = y^{\log x}$. (ii) $x^{\log y - \log x} \times y^{\log x - \log x} \times x^{\log x - \log y} = 1$

- 29. In any right-angled triangle ABC, C being the right angle, show that $R+r=\frac{1}{3}(a+b)$.
- 30. Show how to solve a triangle having given the three perpendiculars from the vertices on the opposite sides.

CHAPTER XVI

GRAPHS OF TRIGONOMETRICAL FUNCTIONS

102. Changes in the Trigonometrical ratios of an angle as the angle increases from 0° to 360°.



Suppose an angle traced out by a revolving line starting from OX, changes gradually from 0° to 360°.

Take a circle with centre O of any radius. It is clear that in determining the trigonometrical ratios of an angle XOP_1 in its different positions, we can keep the hypotenuse OP_1 always the same, equal to the radius of the circle.

(i) Changes in sine.

When the angle N_1OP_1 (= θ say) is zero, its sine is zero. As the angle increases from 0° to 90°, the hypotenuse OP_1 remaining the same, the opposite side P_1N_1 is positive and gradually increases, as is evident by comparing the triangles N_1OP_1 and N_2OP_2 .

Hence, $\sin\theta = \frac{P_1N_1}{OP_1}$ gradually increases, until when $\theta = 90^{\circ}$, P_2N_2 and OP_2 both coincide with OY and $\sin\theta$ attains its greatest value 1.

As θ still further increases, from 90° to 180°, the hypotemuse OP_3 retains the same value, but P_3N_3 remaining positive, now gradually diminishes from OY to zero, and so $\sin \theta$ diminishes from 1 to 0. In the third quadrant, as θ increases from 180° to 270°, P_4N_4 is negative and numerically increases from zero to OY', the hypotenuse remaining always positive and unaltered. Sin θ is therefore negative and numerically increases from 0 to 1; in other words, it diminishes gradually from 0 to -1. In the fourth quadrant, as θ increases from 270° to 360°, P_5N_5 remaining negative numerically diminishes from OY' to 0, and $\sin \theta$ therefore remaining negative numerically diminishes from 1 to 0; in other words, it increases from -1 to 0. The results are therefore as follows:

In the first quadrant, as θ increases from 0° to 90°, $\sin \theta$ increases from 0 to 1.

In the second quadrant, as θ increases from 90° to 180°, $\sin \theta$ diminishes from 1 to 0.

In the third quadrant, as θ increases from 180° to 270°, $\sin \theta$ diminishes from 0 to -1.

In the fourth quadrant, as θ increases from 270° to 360°, $\sin \theta$ increases from -1 to 0.

(ii) Changes in cosine.

In the first quadrant, as the angle XOP_1 increases, ON_1 diminishes, from the value of OX at $\theta = 0^{\circ}$ to the value 0 at $\theta = 90^{\circ}$, and is always positive.

In the second quadrant, as θ goes on increasing from 90° to 180°, ON_3 increases numerically from 0 to OX' but is

negative. In the third quadrant, ON_4 remains negative, but diminishes numerically from OX' to 0. In the fourth quadrant, ON_5 is positive and increases from 0 to OX again.

The hypotenuse remains always positive and is equal to OX or OX' in magnitude.

We thus come to the conclusions:

As θ increases from 0° to 90°,

cos 6 diminishes from 1 to 0.

As θ increases from 90° to 180°,

 $\cos \theta$ diminishes from 0 to -1.

As 6 increases from 180° to 270°,

 $\cos \theta$ increases from -1 to 0.

As 6 increases from 270° to 360°,

 $\cos \theta$ increases from 0 to 1.

(iii) Changes in tangent.

As θ goes on increasing from 0° to 90° in the first quadrant, P_1N_1 increases from 0 to OY and simultaneously ON_1 decreases from OX to 0, both remaining positive; hence,

$$\tan \theta = \frac{P_1 N_1}{O N_1}$$
 increases from the value $\frac{O}{O X} = 0$ to $\frac{O Y}{O} \to \infty$.

In the second quadrant, P_3N_3 diminishes from OY to 0 while ON_3 , becoming negative, numerically increases from 0 to OX'. Hence, $\tan \theta = \frac{P_3N_3}{ON_3}$ is negative but numerically diminishes from ∞ to 0, i.e., increases from $-\infty$ to 0.

Immediately before 90°, $\tan \theta$ is positive and very large, while immediately after 90° $\tan \theta$ is negative and numerically very large. In fact, here, as θ passes through the value 90° from the first to the second quadrant, there is

a sudden break or discontinuity in the value of $\tan \theta$, which suddenly passes from a very large positive value to a very large negative value, *i.e.*, from the positive to the negative side in passing through infinity.

In the third quadrant, both P_4N_4 and ON_4 are negative and P_4N_4 increases numerically from 0 to OY' while ON_4 decreases numerically from OX' to 0. Hence, $\tan\theta = \frac{P_4N_4}{ON_4}$ is positive and increases from 0 to ∞ .

In the fourth quadrant, P_5N_5 is negative and numerically diminishes from OY' to 0 while ON_5 is positive and increases from 0 to OX. Hence, $\tan \theta = \frac{P_5N_5}{ON_5}$ is negative and numerically diminishes from ∞ to 0, *i.e.*, increases from $-\infty$ to 0.

In passing through 270°, there is another discontinuity, tan θ suddenly passing from the positive to the negative side through infinity.

The results are therefore as follows:

As θ increases from 0° to 90°, tan θ increases from 0 to ∞

As θ passes through 90°, $\tan \theta$ suddenly changes from $+\infty$ to $-\infty$

As θ increases from 90° to 180°, tan θ increases from

- ∞ to 0

As θ increases from 180° to 270°, tan θ increases from

0 to ∞

As θ passes through 270°, tan θ suddenly changes from $+\infty$ to $-\infty$

As θ increases from 270° to 360°, tan θ increases from - ∞ to 0.

(iv) Changes in cotangent.

From the changes in the value of the tangent the changes in cot $\theta = \frac{1}{\tan \theta}$ are traced as follows:

0 increasing from 0° to 90°, cot 0 diminishes from

∞ to 0

 θ increasing from 90° to 180°, cot θ diminishes from

0 to $-\infty$

As θ passes through 180°, there is a sudden change in $\cot \theta$ from $-\infty$ to $+\infty$

 θ increasing from 180° to 270°, cot θ diminishes from $+\infty$ to 0

 θ increasing from 270° to 360°, cot θ diminishes from 0 to $-\infty$

As θ passes through 360°, cot θ again suddenly changes from $-\infty$ to $+\infty$.

(v) Changes in secant.

For $\sec \theta = \frac{1}{\cos \theta}$, the results are as follows:

From 0° to 90° for θ , see θ increases from 1 to ∞ .

Here, there is a sudden change from $+\infty$ to $-\infty$.

Then from 90° to 180°, sec θ increases from $-\infty$ to -1.

From 180° to 270° , sec θ diminishes from -1 to $-\infty$.

Here, again there is a sudden change from $-\infty$ to $+\infty$.

Then from 270° to 360°, sec 0 diminishes from ∞ to 1.

(vi) Changes in cosecant.

For cosec $\theta = \frac{1}{\sin \theta}$, the results are as follows:

From 0° to 90°, cosec θ diminishes from ∞ to 1.

From 90° to 180°, cosec & increases from 1 to ...

Here, cosec θ suddenly changes from $+\infty$ to $-\infty$.

Then from 180° to 270°, cosec 0 increases from

 $-\infty$ to -1.

From 270° to 360°, cosec θ diminishes from -1 to $-\infty$. As θ passes through 360°, cosec θ again suddenly

changes from $-\infty$ to $+\infty$.

Note. As θ increases by complete multiples of 2π (i.e., 360°) we know that all the trigonometrical ratios romain unaltered. Hence after 360° , as θ goes on increasing, the same series of values for the ratios are repeated over and over again for each complete revolution of the revolving line. The trigonometrical ratios are therefore all of them periodic functions having the same period 2π ,* after each of which the same cycle of values is repeated.

The changes traced out above, of the trigonometrical ratios, may be much more clearly demonstrated to the eye from a study of their graphs.

103. Graphs of Trigonometrical Functions.

Just like algebraic functions, trigonometrical functions (i.e., $\sin x$, $\cos x$, $\sin^2 2x + \tan \frac{x}{2}$, etc.) may be conveniently represented by means of graphs, showing their changes with the change in the values of the angles.

The method is the same as for graphs in Algebra. Two straight lines XOX' and YOY', intersecting at right angles are taken as axes of co-ordinates. Along the x-axis, the angles are represented on a suitably chosen scale, positive angles along OX', and negative angles along OX'. Along the y-axis the values of the trigonometrical functions corresponding to the angles are represented on a suitably chosen scale, positive values being measured upwards (along OY'), and negative values downwards (along OY'). Thus, the abscissa and ordinate of a point stand respectively for an angle and its trigonometrical function.

^{*} tan θ and cot θ have a period π .

Plotting a number of points in this way and joining them free-hand, we get the required graph of a given trigonometrical function.

104. Graph of sin x or sine-graph.

Let $y = \sin x$.

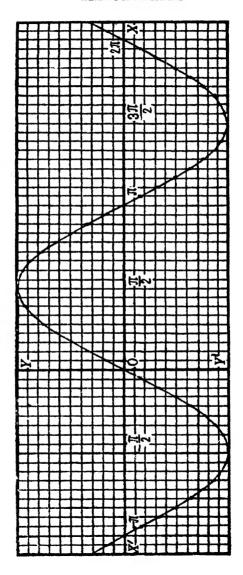
Using the table of natural sines, the corresponding values of x and y are tabulated corresponding to the values of x differing by 10° (the values of y being correct up to two places of decimals) as follows:—

Now, let the scale be so chosen that I small division along OX represents 10°, and 10 small divisions along OY represent unity.*

The points corresponding to the tabulated values are plotted on the graph paper according to the scale chosen and joined free-hand.

The graph is as shown on the next page (drawn here between the range $x = -180^{\circ}$ to $x = +360^{\circ}$).

*According to the graph paper supplied and the range within which the graph is to be drawn, the scale should be suitably chosen in each individual case separately.



ine aph

Note 1. In the table of natural sines, sines of angles from 0° to 90° only are available. With the help of the formula $\sin(-\theta) = -\sin\theta$, $\sin(180^{\circ} - \theta) = \sin\theta$, $\sin(180^{\circ} + \theta) = -\sin\theta$ etc. of Chapter IV, however, the tabulation for $\sin\theta$ shown above, outside the range of 0° to 90° is effected.

Similar is the case of tabulation for other graphs in the following pages.

Note 2. Peculiarities of the sine-graph.

From the figure, the following features will be apparent:—(i) the graph is continuous, and wavy in form; (ii) the maximum value of $\sin x$ is +1 and the minimum value is -1, these values being attained for values of x which are odd multiples of 90° ; (iii) $\sin x$ is 0 at the origin and at points for which x is an even multiple of 90° i.e., any multiple of 180° ; (iv) that $\sin\left(\frac{\pi}{2}-x\right)=\sin\left(\frac{\pi}{2}+x\right)$, $\sin\left(\pi-x\right)=\sin x$, $\sin\left(-x\right)=-\sin x$, $\sin\left(\pi+x\right)=-\sin x$ etc.; (v) since $\sin\left(2\pi\pi+x\right)=\sin x$, the portion between 0 to 2π is repeated over and over again on either side.

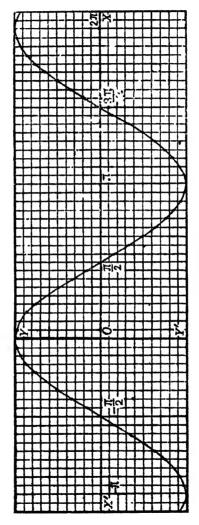
105. Graph of cos x or cosine-graph.

Let $y = \cos x$.

Using the table of natural cosines (see Note 1 of the previous Article), the corresponding values of x and y are tabulated at intervals of 10° for x as follows:—

æ	-90°	- 90°	-70°	− 60°	- 50°	-40°	-30°	- 20°	-10°
y or	0	-17	'84	.20	'64	.77	.87	.94	.98

æ	0.	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	etc.
y or cos x	1	. 98	'94	-87	.77	·6 4	·50	-84	17	0	- 17	·84	etc.



Cosine-graph

Now, choosing the scale such that 1 small division along OX represents 10°, and 10 small divisions along OY represent unity, the points corresponding to the above tabulated values are plotted and joined free-hand.

We then get the required graph, which is shown on the annexed page (shown here between the range $-\pi$ to $+2\pi$ of x).

Note. It is apparent from the figure, that the cosme-graph is exactly the same as the sine-graph only shifted wholesale backwards (to the left) through a space of 90°.

This is due to the fact that $\sin (90^{\circ} + x) = \cos x$, or $\sin x = \cos (x - 90^{\circ})$ so that the ordinate in the sine-graph corresponding to any value of x = the ordinate of the cosine-graph corresponding to a value of x which is 90° less than before.

106. Graph of tan x or tangent-graph,

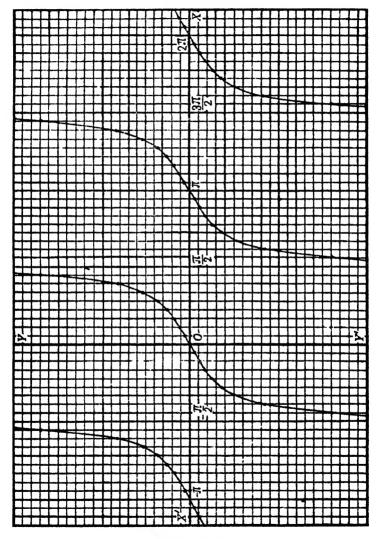
Let $y = \tan x$.

Using the table of natural tangents, the corresponding values of x and y are tabulated at intervals of 10° of x as follows:—

æ	- 20°	-10°)° 10°	20° 30°	40°	50°	60°	70°	80°	90°	100°	etc.	
y or tan x	36	- 18	0 18	.36 .28	.84	1.19	1.78	2.75	5.67	•	- 5.67	etc.	

Now, choosing the scale such that 1 small division along OX represents 10°, and 3 small divisions along OY represent unity, the points corresponding to the above tabulated values are plotted and joined free-hand.

The graph is as shown on the next page (shown here between the range $-\pi$ to $+2\pi$ for x).



Tangent-graph

Note. Peculiarities of the tangent-graph.

From the figure, the following points will be apparent: (i) That the curve is not continuous, but consists of separate branches or portions, the points of discontinuities being the values of x corresponding to the old multiples of $\frac{\pi}{2}$. (ii) As x passes through these points from the left to the right, the value of $\tan x$ suddenly changes from very large positive values on the left to very large negative values on the right. (iii) The lines parallel to y-axis corresponding to the odd multiple of $\frac{\pi}{2}$ are continuously approached by the graph on either side, but never actually meet. Such lines are called asymptotes to the curve. (iv) Since $\tan (n\pi + x) = \tan x$, each branch is simply a repetition of the branch from $-\frac{\pi}{6}$ to $+\frac{\pi}{6}$.

107. Graph of cot x or cotangent-graph.

As before the values of x and y (= cot x) are tabulated, and with the same scale as in the tangent-graph the points are plotted and joined free-hand

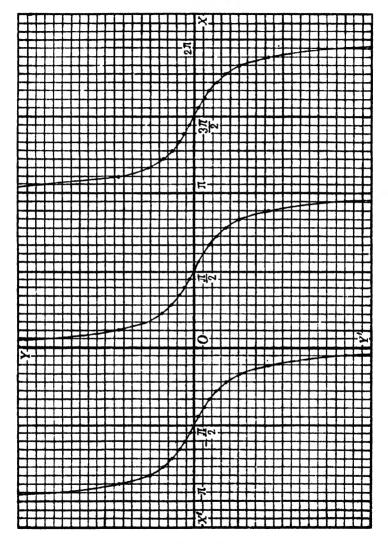
The graph is shown on the next page (between $x = -\pi$ to $x = +2\pi$).

This graph also, like the tangent-graph, is discontinuous, the points of discontinuity being x=0 and $x=n\pi$. The portion between x=0 and $x=\pi$ is repeated over and over again on either side, as is consequent from the formula $\cot (n\pi + x) = \cot x$.

108. Graph of cosec x or cosecant-graph.

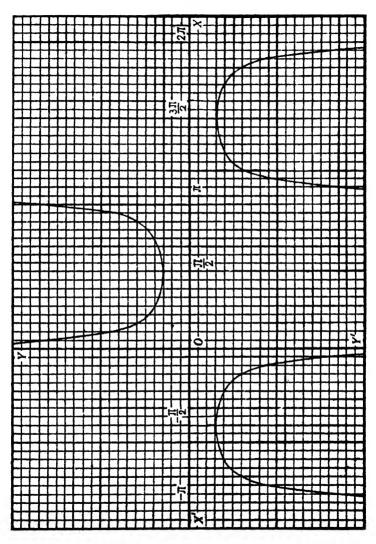
The corresponding values of x and y are tabulated at intervals of 10° of x as follows:—

-	-20°	-10°	0°	10°	20°	30°	ete.	80°	90°	10 0°	110°	etc.	-
y or cosec x	-2.03	-5.76	00	5.76	2.25	2	etc.	1.02	1	1.03	1.06	etc.	



Cotangent-graph

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Cosecant-graph

[If the table of natural cosecants be not available, the table of natural sines may be used and the values of cosec x

 $=\frac{1}{\sin x}$ may be calculated for different values of x.

The scale is so chosen that 1 small division along OX represents 10°, and 3 small divisions along OY represent unity.

The tabulated points are now plotted and joined free-hand.

The graph is shown on the previous page (between the range $x = -\pi$ to $x = 2\pi$).

Note. This graph also consists of detached branches, the points of discontinuity being x=0 and $x=n\pi$. The value of y never lies between ± 1 , being always greater than 1 or less than -1. The lines $x=n\pi$ are asymptotes. The portion between x=0 to $x=2\pi$ is repeated on either side, over and over again.

109. Graph of sec x or secant-graph.

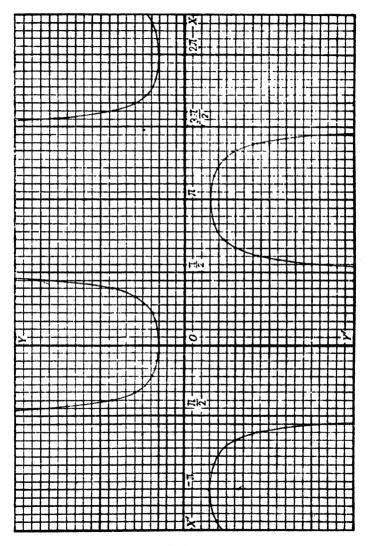
The corresponding values of x and y ($=\sec x$) are tabulated as in the case of cosecant-graph, by making use of the table of cosines, if a table of secants be not available.

With the same scale as in the cosecant-graph, the tabulated points have been plotted and joined free-hand.

The graph is shown in the adjoining page (between the range $x = -\pi$ to $x = 2\pi$).

Note. It is apparent from the figure that the secant-graph is exactly the same as the cosecant-graph, only shifted backwards (to the left) through a space of 90°.

This is due to the fact that cosec $(90^{\circ} + x) = \sec x$. [See note below Art. 105]



Secant-graph

110. Graphs of other Trigonometrical Expressions.

Graphs of other trigonometrical functions may be obtained in a similar manner.

We illustrate this by an example.

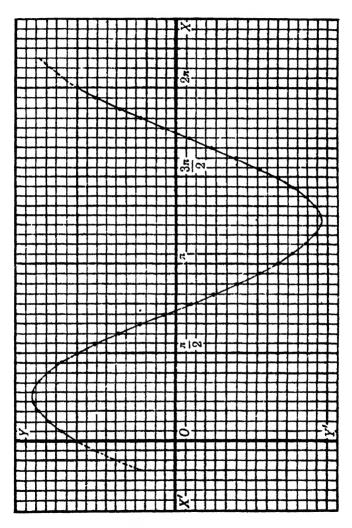
Ex. Draw the graph of $y = \sin x + \cos x$ between the range x = 0 to $x = 2\pi$, and find from the graph the values of x for which (i) y = 0, (ii) y is maximum, (iii) y is minimum.

[C. U. 1934]

From the table of natural sines and cosines, corresponding to each value of x, the values of $\sin x$ and $\cos x$ may be separately obtained and then added to give y; or else we may write $y = \sin x + \cos x = \sqrt{2} (\sin x \cos \frac{1}{4}\pi + \cos x \sin \frac{1}{4}\pi) = \sqrt{2} \sin (x + \frac{1}{4}\pi)$, and corresponding to any value of x, $\sin (x + \frac{1}{4}\pi)$ may be deduced from the sine-table, and this multiplied by $\sqrt{2}$ (=1'414) will give y.

The corresponding values of x and y are tabulated at intervals of 10° of x, between the interval x = 0 to $x = 2\pi$ as follows:—

GRAPHS 207



Graph of $\sin x + \cos x$

æ	210°	220°	230°	240°	250°	260°	270°	280°
y	-1.37	-1.41	-1.41	-1.37	-1'27	-1.12	-1	81
x	290°	300°	310°	320°	330°	840°	850°	360°
y	29	- '37	-:13	'13	'37	.29	'81	1

The scale is chosen so that 1 small division along OX represents 10°, and 10 small divisions along OX represent unity.

The tabulated points are now plotted and joined. The graph is as shown on the previous page.

From the graph we find that (i) y = 0 when $x = 135^{\circ}$, and $x = 315^{\circ}$, (ii) y is maximum when $x = 45^{\circ}$, (iii) y is minimum when $x = 225^{\circ}$.

Graphical solution of Equations.

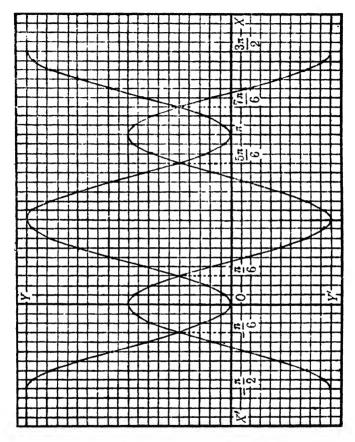
Trigonometrical equations, just like algebraic equations may be solved graphically. In fact in many practical cases, particularly where the solutions are not obtained in terms of the standard angles, the graphical method of solution is the only one which is found convenient and is actually adopted. The method is illustrated by the following examples.

Ex. 1. Solve graphically the equation $2 \sin^3 x = \cos 2x$, giving only those solutions of x which lie between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

[C. U. 1938. '46. '48]

We draw two graphs, namely
$$y = 2 \sin^3 x = (1 - \cos 2x)$$
 and $y = \cos 2x$

by tabulating the corresponding values of x and y for the two cases separately, making use of the table of natural



Graphical solution of $2 \sin^2 x = \cos 2x$.

cosines, for the range $x = -\frac{\pi}{2}$ to $x = \frac{3\pi}{2}$, at intervals of 10° or 15° of x.

With the same scale, namely, 1 small division along OX representing 10° , and 10 small divisions along OY representing unity, we plot the tabulated values for the two cases in the same graph paper, and joining them, we get the two graphs, as shown in the adjoining page.

We find that the two graphs intersect, and thus have the same abscissæ and ordinates at the points for which

$$x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}$$

Thus, $2\sin^2 x = \cos 2x$ is satisfied for the values of x given by

$$x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6} \text{ and } \frac{7\pi}{6}$$

which are the required solutions within the range

$$-\frac{\pi}{2}$$
 to $\frac{3\pi}{2}$.

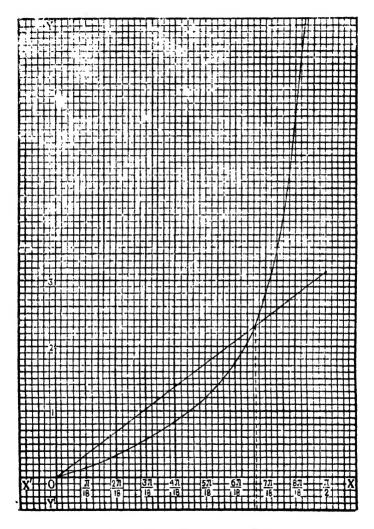
Ex. 2. Solve graphically the equation $\tan x = 2x$ between x = 0 and $x = \frac{\pi}{2}$. [C. U. 1939]

Here, x is supposed to be measured in radians.

First of all we draw separately two graphs, namely

$$y = 2x$$
 ··· (i)

and
$$y = \tan x$$
 ... (ii)



Graphical solution of $\tan x = 2x$.

The corresponding values of x and y within the range x=0 and $x=\frac{\pi}{2}$ are tabulated in case (i) as follows:

(in radians)	0	π δ	$\frac{\pi}{3}$	# 2
y (i.e. 2x) (numerical value)	0	1.05	2.10	3.12

and in case (ii) as follows:

æ (in radians)	0	π 18	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	4π 18	5π 18	6π 18	$\frac{7\pi}{18}$	8π 18	π 2
y (i.e. tan x) (numerical value)	0	18	.36	-57	.84	1.10	1.78	2.75	5.67	00

Now, choosing the same scale, namely 5 small divisions along OX to represent $\frac{\pi}{18}$ radians, and 10 small divisions along OY to represent unity, we plot the tabulated points for the two cases in the same graph paper and joining them we get the two graphs within the range x=0 and $x=\frac{\pi}{2}$ as shown in the adjoining page.

We find that the two graphs intersect at the point given by x=0 and also at the point corresponding to 33.5 small divisions along OX, which, from our chosen scale, represents $x=\frac{33.5}{5}\times\frac{\pi}{18}$ radians = 1.17 radians (approximately).

Hence, the given equation $\tan x = 2x$ is satisfied between x = 0 and $x = \frac{\pi}{2}$ by the values of x, namely x = 0 and x = 1.17 (approximately), which are the required solutions in radians.

Examples XVI

- 1. Draw the graphs of :
 - (i) $\sin 3x$ between $x = 0^{\circ}$ to $x = 180^{\circ}$.
 - (ii) $\tan \frac{3}{2}x$ between $x = -\frac{1}{2}\pi$ to $x = \pi$.
 - (iii) $\sin \theta \cos \theta$ between $\theta = -\pi$ to $\theta = +\pi$.
 - (iv) $\frac{1}{\cos^2 \theta \sin^2 \theta}$ between $\theta = -\frac{\pi}{2}$ to $+\frac{\pi}{2}$.
 - (v) $\cos (\pi \sin x)$ between x = 0 to $x = \frac{1}{2}\pi$.
 - (vi) $\sin \theta \sqrt{3} \cos \theta$ between $\theta = 0$ to $\theta = \pi$.
 - (vii) $\frac{1}{2}$ cosec $\frac{1}{2}x$ between x=0 to $x=2\pi$.
- 2. (i) Trace the changes in the sign of $\cos \theta \sin \theta$ as θ changes from 0° to 360° . Verify your conclusions by a graph.
- (ii) Trace the changes in sign and magnitude of $2 \sin \theta \sin 2\theta$. [B. H. U. 1931]
- 3. Draw the graph of $y = \sin(x + \frac{1}{2}\pi)$ between the limits $x = -\pi$ and $x = +\pi$.
- 4. Draw the graphs of $\sin \theta$ and $\cos \theta$ between $\theta = 0$ and $\theta = \pi$. Find the points where the graphs intersect.

[C. U. 1936, '46]

5. Construct the graphs of $\tan x$ and $\cos x$ between 0 and $\frac{1}{2}\pi$ for x, making a tabulation of the values of y dividing the interval into 9 equal parts.

If $\tan x = \cos x$, find approximately the value of x from the above two graphs. [C. U. 1943]

6. Obtain graphically a solution of the equation $\tan x = 1$, between x = 0 and $x = \frac{1}{2}\pi$. [C. U. 1937]

[Draw the graphs of $y = \tan x$ and y = 1.]

- 7. Draw the graph of $\cos x \sin 2x$ for values of x lying between 0° and 90° and hence obtain the least value of $\cos x \sin 2x$ in this range.
 - 8. Solve graphically the equations:
 - (i) $x \tan x = 0$, between x = 0 and $x = \frac{1}{2}\pi$.

[C. U. 1945]

- (ii) $5 \sin \theta + 2 \cos \theta = 5$, between $\theta = 0^{\circ}$ and $\theta = 270^{\circ}$.
- [Draw the graphs of y=5 sin $\theta+2\cos\theta$ and y=5 and find the common points.] [C. U. 1947]
 - (iii) cot θ tan θ = 2, between θ = 0 and θ = π .

[C. U. 1949]

- (iv) cosec $x = \cot x + \sqrt{3}$, between x = 0 and x = n.
- (v) $\cos x = \sin 2x + \frac{1}{2}$, between $x = -\frac{1}{2}\pi$ and $x = +\frac{1}{2}\pi$.
- (vi) $5 \tan x = 2x$, between 0 and 2π .
- (vii) $2 \sin x + x 3 = 0$.
- (viii) $x^2 = \cos x$.
- (ix) $x = \cos^2 x$.

[Draw the graphs of $y = \cos 2x$ and y = 2x - 1.]

- 9. Represent by a graph the displacement given by $s=2 \sin t + \sin 3t$.
- 10. Show graphically that the equation $2 \sin x + \cos 2x = \frac{1}{2}x$ has only three real roots.
 - 11. Sketch the graphs:

y=x, $y=\sin x$, $y=\tan x$, in $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$. From the nature of graphs near the origin, can you suggest any relation among them at the origin? [C. U. 1952]

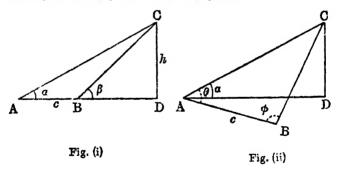
CHAPTER XVII

MISCELLANEOUS THEOREMS AND EXAMPLES

Sec. A

HARDER PROBLEMS ON HEIGHTS AND DISTANCES

- 112. Some simple practical applications of Trigonometry, dealing with easy problems on determination of heights and distances, have already been discussed in Chapter V. The problems in the present section are of a more general character, requiring for their solutions, the general relations between the sides and angles of a triangle, as also some geometrical skill.
- 113. To find the height and the distance of an inaccessible object standing on a horizontal plane.



Let CD be the object, which is observed from a point A on a horizontal ground, α being the observed elevation of its top C. Let h be the required height CD and d the required distance AD of the object from A.

Case I. If possible, measure off any suitable distance AB (=c) from A directly towards the object, and from B let the observed elevation of C be β .

Then, from fig. (i),

$$c = AD - BD = h \cot a - h \cot \beta$$
$$h \begin{pmatrix} \cos a - \cos \beta \\ \sin a - \sin \beta \end{pmatrix} = h \sin (\beta - a)$$
$$\sin a \sin \beta$$

 $\therefore h = c \sin a \sin \beta \csc (\beta - a).$

Also $d = AD = h \cot a = c \cos a \sin \beta \csc (\beta - a)$.

Note. Each of the above expressions for h and d is in a suitable form for logarithmic computation.

Case II. If however it is not convenient to measure the length AB directly towards the object, we may proceed as follows.

Measure off the length AB(=c) in any suitable direction from A. From A let the observed elevation of C be a as before. The angles CAB and CBA are also observed from A and B respectively. Let these be θ and ϕ .

We get from fig. (ii) in this case,

in
$$\triangle ABC$$
, $\frac{AC}{\sin \phi} = \frac{AB}{\sin C}$

i.e.,
$$= \frac{c}{\sin(180^{\circ} - \theta - \phi)} = \frac{c}{\sin(\theta + \phi)}$$

 $\therefore AC = c \sin \phi \csc (\theta + \phi).$

$$h = AC \sin a = c \sin a \sin \phi \csc (\theta + \phi),$$

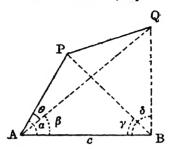
and $d = AD = AC \cos \alpha = c \cos \alpha \sin \phi \csc (\theta + \phi)$.

Note. Here also, the expressions for h and d are suitable for calculation by logarithm.

114. To find the distance between two visible but inaccessible objects.

Let P and Q be the objects whose distance apart is required.

Take two suitable points A and B for observation, the distance between which is measured, say c.



At A, measure the angles PAQ, PAB, and QAB (the second observation being unnecessary if all the four points P, A, B, Q are in the same plane, for in that case, $\angle PAB = \angle PAQ + \angle QAB$). Let these be θ , α and β respectively.

At B, measure the angles PBA and QBA, and let them be γ and δ .

From
$$\triangle PAB$$
, $\frac{PA}{\sin \gamma} = \frac{c}{\sin (180^{\circ} - \alpha - \gamma)} = \frac{c}{\sin (\alpha + \gamma)}$, whence $PA = c \sin \gamma$ cosec $(\alpha + \gamma)$.

Similarly, from $\triangle QAB$,

$$QA = c \sin \delta \csc (\beta + \delta).$$

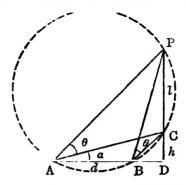
Lastly, from $\triangle PAQ$,

$$PQ^2 = PA^2 + QA^2 - 2PA \cdot QA \cdot \cos \theta.$$

Thus, PQ is determined.

- 115. Some more illustrative examples of harder problems on heights and distances are worked out helow.
- Ex. 1. A flagstaff is fixed on the top of a tower standing on a horizontal plane. An observer walking directly towards the foot of the tower, observes the angle subtended

by the flagstaff from two positions on his path to be the same namely 0. The distance between these two positions is d, and the angle subtended by the tower at his first position is a. Find the height of the tower and the length of the flagstaff.



Let CD be the tower, PC the flagstaff, whose heights required are h and l respectively. A and B are the points of observation.

 \therefore $\angle PAC = \angle PBC = \theta$, the points P, A, L, C are concyclic.

$$\angle CBD = \angle APC = 90^{\circ} - \angle PAD = 90^{\circ} - (\theta + \alpha).$$
Now,
$$d = AD - BD = h \cot \alpha - h \cot (CBD)$$

$$= h \left\{ \cot \alpha - \tan (\theta + \alpha) \right\}$$

$$= h \left\{ \frac{\cos \alpha}{\sin \alpha} - \frac{\sin (\theta + \beta)}{\cos (\theta + \alpha)} \right\} = h \frac{\cos (\theta + 2\alpha)}{\sin \alpha \cos (\theta + \alpha)}$$

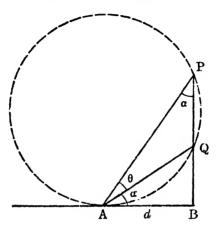
 $\therefore h = d \sin a \cos (\theta + a) \sec (\theta + 2a).$

Again, from $\triangle APC$,

$$\frac{l}{\sin \theta} \frac{AC}{\sin APC} \frac{h}{\sin a \cos (\theta + a)} \frac{d}{\cos (\theta + 2a)}$$

$$\therefore \quad l = d \sin \theta \sec (\theta + 2a).$$

Ex. 2. A man walking towards a laulding, on which a flagstaff is fixed, observes the angle subtended by the flagstaff to be greatest, when he is at a distance d from the building. If θ be the observed greatest angle, find the length of the flagstaff and the height of the building.



Let QB be the building, and PQ the flagstaff. It is easily seen from Geometry that the point of contact A of a circle through Γ and Q touching the path of the observer on the ground, is the point at which the angle subtended by PQ is the greatest.

Thus,
$$\angle QAB = \angle APQ = a$$
 say.

Then, $\angle PAB + \angle APB = 90^{\circ}$ or, $\theta + 2a = 90^{\circ}$ (i)

Now, $PQ = PB - QB = d \tan (\theta + a) - d \tan a$

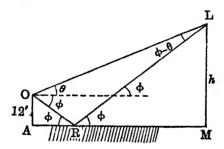
$$= d \frac{\sin (\theta + a)}{\cos (\theta + a)} \frac{\sin a}{\cos a}$$

$$= d \frac{\sin \theta}{\cos (\theta + a) \cos a} \frac{2d \sin \theta}{\cos (\theta + 2a) + \cos \theta}$$

$$= 2d \tan \theta. \qquad [from (i)]$$
Also, $QB = d \tan a = d \tan (\frac{\pi}{4} - \frac{\theta}{2})$.

Ex. 3. The angle of elevation of a light at the top of a distant tower from a point 12 ft. above a lake is 24° 55', and the angle of depression of its reflection in the lake is 35° 5'. Find the height of the tower correct to two decimal places, having given

log 2 = '30103, log 3 = '47712
log 588 = 2'76938, log 589 = 2'77012
$$L \sin 10^{\circ} 10' = 9'24677$$
.



Let L be the light at the top of the tower LM, LRO the ray from L, which reflected in the lake at R, reaches the observer O, so that OR is the direction in which the reflexion is seen, and thus from the laws of reflexion, $\angle ORA = \angle LRM = \phi$ (say) which is evidently equal to the angle of depression of the reflexion, i.e., 35° 5'.

Let θ denote the angle of elevation of L from O, i.e., 24° 55'.

Now, from the figure, in $\triangle ORL$,

$$RI_{\sin}(\theta + \phi) = \frac{OR}{\sin(\phi - \theta)} = \frac{12}{\sin\phi\sin(\phi - \theta)} \text{ ft.}$$

$$h = LM = RI_{c}\sin\phi = 12 \frac{\sin(\theta + \phi)}{\sin(\phi - \theta)} = 12 \frac{\sin 60^{\circ}}{\sin(10^{\circ} 10')}$$

$$= \frac{6\sqrt{3}}{\sin(10^{\circ} 10')} = \frac{2.3^{\frac{3}{2}}}{\sin(10^{\circ} 10')}.$$

Hence,
$$\log h = \log (2.3^{\frac{8}{2}}) - \log \sin (10^{\circ} 10')$$

= $\log 2 + \frac{8}{2} \log 3 + 10 - L \sin (10^{\circ} 10')$
= $30103 + \frac{8}{2} (47712) + 10 - 9.24677$
= 1.76994 .

From the given data, it is seen that

 $\log h$ lies between $\log 58'8$ and $\log 58'9$.

Hence, if
$$h = 58.8 + x$$
, diff. for $1 = 1.77012 - 1.76938$
= '00074,

and diff. for x = 1.76994 - 1.76938 = .00056.

... by the theory of proportional parts.

$$\frac{x}{1} = \frac{56}{74} = .75$$
; $\therefore x = .075 = .08$ approximately.

Thus. h = 58.88 ft.

Examples XVII(a)

- 1. The angles of elevation of the top of a plam tree standing on horizontal ground, observed from two points A and B, distant 40 and 30 feet from the foot, and in the same straight line with it are found to be complementary. Prove that the height of the tree is nearly 35 feet, and that the angle subtended at the top of the tree by the line AB is $\sin^{-1}\frac{1}{4}$.
- 2. The angles of elevation of an aeroplane from two places one mile apart and from a point half way between them are found to be 60°, 30° and 45° respectively. Show that the height of the aeroplane $440 \sqrt{6}$ yards.
- 3. A building with ten storeys, each of equal height x ft., stands on one side of a wide street, and from a point on

the other side of the street directly opposite to the building, it is observed that the three uppermost storeys together and the two lowest storeys together subtend equal angles. Show that the width of the street is $x\sqrt{140}$ ft.

- 4. A two-storeyed building has the height of its lower storey 12 ft. and that of the upper storey 13 ft. Find at what distance the two storeys subtend equal angles to an observer's eye at a height 5 feet from the ground.
- 5. A vertical rod is erected in a horizontal rectangular field ABCD. The angular elevation of its top from A, B, C, D are α , β , γ , δ . Show that

$$\cot^2 a - \cot^2 \beta = \cot^2 \delta - \cot^2 \gamma.$$

6. The angles of elevation of a bird flying in a horizontal straight line, from a fixed point at four successive observations are α , β , γ , δ , the observations being taken at equal intervals of time. Assuming that the speed of the bird is uniform, show that

$$\cot^2 \alpha - \cot^2 \delta = 3(\cot^2 \beta - \cot^2 \gamma).$$

7. A man on a hill observes that three towers on a horizontal plane subtend equal angles at his eye and that the angles of depression of their bases are a, β , γ . If a, b, c are the heights of the towers, prove that

$$\frac{\sin (\beta - \gamma)}{a \sin a} + \frac{\sin (\gamma - a)}{b \sin \beta} + \frac{\sin (a - \beta)}{c \sin \gamma} = 0.$$

8. A gun is fired from a fort F at a distance d from a station O, and from two stations A and B in a straight line with O and distant a and b respectively from O, the intervals between seeing the flash and hearing the report are t and t'. Show that the velocity of sound is

$$\sqrt{\frac{(d^2-ab)(a-b)}{at'^2-bt^2}}.$$

9. A person observes the elevation of the top of a telegraph post which is E. S. E. of him to be 45°, and at noon, the extremity of its shadow is to the N. E. of him; if the length of the shadow be x, show that the height of the post is $x\sqrt{2-\sqrt{2}}$.

10. A straight tree on the horizontal ground leans towards the North; at two points due South and distant a, b respectively from the foot, the angular elevations of the top of the tree are a and β . Show that the inclination of the tree to the horizon is

$$\tan^{-1}\left(\frac{a-b}{a \cot \beta - \cot a}\right).$$

11. An observer on a carriage moving with a speed V along a straight road observes in one position that two distant trees are in the same line with him which is inclined at an angle θ to the road. After a time t, he observes that the trees subtend their greatest angle ϕ ; show that the distance between the tree is

$$2Vt \sin \theta \sin \phi / (\cos \theta + \cos \phi).$$

12. A train travelling on one of two straight intersecting railways subtends at a certain station on the other line, angles a and β , when the front of the first carriage and the end of the last carriage reach the junction respectively. Show that the angle of intersection of the two lines is

$$\tan^{-1}\frac{2\sin a\sin \beta}{\sin (a \sim \beta)}.$$

13. Two vessels are sailing in parallel directions, and at one instant the bearing of one from the other a° N. of E. After an hour's sailing the bearing of the first from the second is β° N. of E., and after another hour the bearing is γ° N. of E. Show that the vessels are sailing in a direction θ° N. of E., where

$$\sin (\alpha - \theta) \sin (\gamma - \beta) = \sin (\beta - \alpha) \sin (\gamma - \theta).$$

14. A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground; if the longest shadow it can cast is $3\frac{1}{3}$ times the length of the rod, calculate the altitude of the sun, having given

$$\log 3 = 47712$$
, $L \cos 72^{\circ} 32' 30'' = 9'47712$.

15. A ship sailing N. E. is, at a particular moment, in a line with two light-houses, one of which is situated 5 miles

- due N. of the other. In 3 minutes and also in 21 minutes the light-houses are found to subtend a right angle at the ship. Prove that the ship is sailing at the rate of 10 miles an hour, and that the light-houses subtend their greatest angle at the ship at the end of $3\sqrt{7}$ minutes.
- 16. A parachute was observed in the N. E. at the elevation 45° ; ten minutes afterwards it was found to be due N. at an elevation $22\frac{1}{2}^{\circ}$. The parachute was descending at the rate of 6 miles per hour, and was all along drifted uniformly towards the West by the wind. Show that wind blows at the rate of 6 miles per hour.
- 17. A person wishing to determine the height of a distant temple observes the elevation of its top from a point on the horizontal ground through its base to be 30°. On walking a distance $80 \sqrt{3}$ ft. in a certain direction, he finds the elevation of the top to be the same as before, and then on walking a distance 80 ft. at right angles to the former direction, the elevation is found to be 45°. Show that the height of the temple is 80 ft.
- 18. The shadow of a telegraph post is observed to be half the known height of the post, and sometime afterwards it is equal to the known height; how much will the sun have gone down interval, given

 $\log 2 = 30103$, L tan 63° 26' = 10'3009994 and diff. for 1' = 3159.

19. The side of a hill faces due S., and is inclined to the horizon at an angle a. A straight railway upon it is inclined at an angle β to the horizon; show that the bearing of the railway is

 \cos^{-1} (cot a tan β) E. of N.

20. A spherical time-ball of diameter d at the top of a tower subtends an angle 2a at a point on the ground from which the elevation of its centre is θ ; prove that the height of the centre of the ball above the ground is $\frac{1}{2}d$ sin θ cosec a.

Sec. B-SUMMATION OF FINITE SERIES

116. Method of Difference.

When the rth term of a trigonometrical series can be expressed as the difference of two quantities, one of which is the same function of r as the other is of (r+1), the sum of the series may be readily found as illustrated in the Examples 1 and 2 below.

Ex. 1. Find the sum of the series:

(i) cosec
$$\theta$$
 + cosec 2θ + cosec $2^2\theta$ + \cdots + cosec $2^{n-1}\theta$.

(ii)
$$\frac{\sin x}{\sin 2x \sin 3x} + \frac{\sin x}{\sin 3x \sin 4x} + \frac{\sin x}{\sin 4x \sin 5x} + \cdots to n terms.$$

(i) We have cosec
$$\theta = \frac{1}{\sin \theta} = \frac{\sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta \sin \theta}$$

$$= \frac{\sin (\theta - \frac{1}{2}\theta)}{\sin \frac{1}{2}\theta \sin \theta}$$

$$= \sin \theta \cos \frac{1}{2}\theta - \cos \theta \sin \frac{1}{2}\theta$$

$$= \sin \frac{1}{2}\theta \sin \theta$$

$$= \cot \frac{1}{2}\theta - \cot \theta.$$

Thus, $\csc \theta = \cot \frac{1}{2}\theta - \cot \theta$.

Similarly, cosec $2\theta = \cot \theta - \cot 2\theta$. cosec $2^2\theta = \cot 2\theta - \cot 2^2\theta$.

$$\csc 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta.$$

... by addition, the required sum

$$=\cot \frac{1}{2}\theta - \cot 2^{n-1}\theta.$$

(ii) Here, rth term

$$= \frac{\sin x}{\sin (r+1) x \sin (r+2) x}$$

$$= \frac{\sin \{(r+2) - (r+1)\} x}{\sin (r+1) x \sin (r+2) x}$$

$$= \frac{\sin (r+2)x \cos (r+1)x - \cos (r+2)x \sin (r+1)x}{\sin (r+1)x \sin (r+2)x}$$

$$= \cot (r+1)x - \cot (r+2)x.$$

Putting r=1, 2, 3,...n and adding, the sum of the required series would be found to be

$$\cot 2x - \cot (n+2)x.$$

Ex. 2. Find the sum of the series

$$tan^{-1} \frac{x}{1+1.2x^2} + tan^{-1} \frac{x}{1+2.3x^2} + \cdots$$

+ $tan^{-1} \frac{x}{1+n(n+1)x^2}$

Here, rth term =
$$\tan^{-1} \frac{x}{1+r(r+1)x^2}$$

= $\tan^{-1} \frac{(r+1)x-rx}{1+(r+1)x.rx}$
= $\tan^{-1} \frac{(x+1)x-\tan^{-1} rx}{1+rx}$.

 \therefore putting $r=1, 2, 3, \ldots, n$, we have

$$\tan^{-1} \frac{x}{1+1.2x^2} = \tan^{-1} 2x - \tan^{-1} x$$

$$\tan^{-1} \frac{x}{1+2.3x^2} = \tan^{-1} 3x - \tan^{-1} 2x$$

$$\tan^{-1} \frac{x}{1 + n(n+1)x^2} = \tan^{-1} (n+1)x - \tan^{-1} nx.$$

by addition, the required sum $= \tan^{-1} (n+1)x - \tan^{-1}x,$

- 117. Sometimes the rth term of a series, when multiplied by a factor, can be expressed as the difference of two quantities one of which is the same function of r as the other is of (r+1). It is illustrated in the following two cases.
 - (I) Sum of sines of n angles in A.P.

Let the angles in A.P. be a, $a + \beta$, $a + 2\beta$,... $\{a + (n-1)\beta\}$, the first term being a, and the common difference, β .

Let S denote the sum of the series.

$$\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \cdots + \sin \{\alpha + (n-1)\beta\}.$$

Multiplying each term of the above series by

2 sin (half difference) i.e., by
$$2 \sin \frac{1}{2}\beta$$
, we have

$$2 \sin \alpha \sin \frac{1}{2}\beta = \cos \left(\alpha - \frac{1}{2}\beta\right) - \cos \left(\alpha + \frac{1}{2}\beta\right)$$

$$2 \sin (\alpha + \beta) \sin \frac{1}{2}\beta = \cos (\alpha + \frac{1}{2}\beta) - \cos (\alpha + \frac{3}{2}\beta)$$

$$2 \sin (a + 2\beta) \sin \frac{1}{2}\beta = \cos (a + \frac{5}{2}\beta) - \cos (a + \frac{5}{2}\beta).$$

$$2 \sin \{a + (n-1)\beta\} \sin \frac{1}{2}\beta$$

$$= \cos \left(a + \frac{2n-3}{2}\beta\right) - \cos \left(a + \frac{2n-1}{2}\beta\right)$$

Adding vertically, we have

$$2 \sin \frac{1}{2}\beta \cdot S = \cos\left(a - \frac{\beta}{2}\right) - \cos\left(a + \frac{(2n-1)}{2}\beta\right)$$
$$= 2 \sin\left(a + \frac{n-1}{2}\beta\right) \sin\frac{n\beta}{2}.$$

$$\therefore S = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \sin \left(\alpha + \frac{n-1}{2}\beta\right).$$

Cor. Putting $\beta = a$, we get $\sin a + \sin 2a + \sin 3a + \cdots + \sin na$

$$=\frac{\sin\frac{na}{2}}{\sin\frac{a}{2}}\sin\frac{n+1}{2}a$$

(II) Sum of cosines of n angles in A.P.

As before, let S denote the sum of the series

$$\cos u + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \cdots + \cos \{\alpha + (n-1)\beta\}.$$

Multiplying each term of the above series by

2 sin (half difference), we have

$$2 \cos a \cdot \sin \frac{1}{2}\beta = \sin (a + \frac{1}{2}\beta) - \sin (a - \frac{1}{2}\beta)$$

$$2\cos(\alpha+\beta)\cdot\sin\tfrac{1}{2}\beta=\sin(\alpha+\tfrac{3}{2}\beta)-\sin(\alpha+\tfrac{1}{2}\beta)$$

$$2\cos\left(\alpha+2\beta\right).\sin\frac{1}{2}\beta=\sin\left(\alpha+\frac{5}{2}\beta\right)-\sin\left(\alpha+\frac{3}{2}\beta\right)$$

.....

2 cos
$$\{a + (n-1)\beta\}$$
 . sin $\frac{1}{2}\beta$
= sin $\left(a + \frac{2n-1}{2}\beta\right)$ - sin $\left(a + \frac{2n-3}{2}\beta\right)$.

Adding vertically, we have

$$2 \sin \frac{1}{2}\beta.S = \sin \left(a + \frac{2n-1}{2}\beta\right) - \sin \left(a - \frac{\beta}{2}\right)$$
$$= 2 \cos \left(a + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}.$$

$$\therefore S = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \left(\alpha + \frac{n-1}{2}\beta\right).$$

Cor. Putting $\beta = a$, we get

$$\cos a + \cos 2a + \cos 3a + \cdots + \cos na = \frac{\sin \frac{na}{2}}{\sin \frac{a}{0}} \cos \frac{n+1}{2}a.$$

Note. The sum of the cosine series may be deduced from that of the sine series by writing $a + \frac{\pi}{2}$ for a.

As an aid to memory, the two formulæ of this article may be expressed in language as follows:

since,
$$a + \frac{n-1}{2}\beta = \frac{a+a+(n-1)\beta}{2}$$
,

Sum of sines of n angles in A.P.

$$= \frac{\sin \frac{n \cdot diff}{2}}{\sin \frac{diff}{2}} \sin \frac{first \ angle + last \ angle}{2}$$

Sum of cosines of n angles in A.P.

$$= \frac{\sin n. diff.}{2} \cos \frac{first \ angle + last \ angle}{2}$$

Note. From the above formulæ, it is clear that if $\sin \frac{n\beta}{2} = 0$, then the sum of the sine series as also the sum of the cosine series is zero.

Now, if $\sin \frac{n\beta}{2} = 0$, then $\frac{n\beta}{2} = k\pi$, or $\beta = \frac{2k\pi}{n}$, where k is an integer.

Thus, the sum of the sines and the sum of the cosines of n angles in A.P. are each equal to serv when the common difference of the angles is an even multiple of $\frac{\pi}{}$.

Ex. 1. Find the sum of n terms of the series $\sin \alpha - \sin (\alpha + \beta) + \sin (\alpha + 2\beta) - \cdots$

Since, $\sin (\pi + \theta) = -\sin \theta$; $\sin (2\pi + \theta) = \sin \theta$ etc.

... the series is equal to

 $\sin a + \sin (\pi + \alpha + \beta) + \sin (2\pi + \alpha + 2\beta) + \cdots$

i.s., equal to a series in which the common difference of the angles is $\beta + \pi$ and the last angle is $\alpha + (n-1)(\beta + \pi)$.

$$\frac{\sin\frac{n(\beta+\pi)}{2}}{\sin\frac{\beta+\pi}{2}}\cdot\sin\left\{\alpha+\frac{(n-1)(\beta+\pi)}{2}\right\}.$$

Ex. 2. Find the sum of the series
$$\sin^2 \theta + \sin^2 2\theta + \sin^2 3\theta + \dots + \sin^2 n\theta.$$

Since, $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$, $\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$, etc.

... the given series

$$= \frac{1}{2}(1 - \cos 2\theta) + \frac{1}{2}(1 - \cos 4\theta) + \dots + \frac{1}{2}(1 - \cos 2n\theta)$$

$$= \frac{n}{2} - \frac{1}{2}(\cos 2\theta + \cos 4\theta + \dots + \cos 2n\theta)$$

$$= \frac{n}{2} - \sin n\theta$$

$$= \frac{n}{2} - \frac{1}{2} \frac{\sin n\theta}{\sin \theta} \cos (n+1)\theta.$$
 [by Art. 117]

Ex. 3. Sum the series

$$\cos a + 2 \cos (a + \beta) + 3 \cos (a + 2\beta) + \cdots$$

 $\cdots + n \cos \{a + (n-1)\beta\}.$

Let u_r denote the rth term and S denote the sum of the given series.

Now,
$$2 \cos \beta \cdot u_r = 2 \cos \beta \cdot r \cos \{a + (r-1)\beta\}$$

= $r [\cos (a + r\beta) + \cos \{a + (r-2)\beta\}]$.

... putting r=1, 2, 3,...n and adding together,

we get $2\cos\beta$. S.

Now, subtract $2 \cos \beta$. S from 2S; then

$$2S(1-\cos\beta) = (n+1)\cos\{a+(n-1)\beta\}$$
$$-\cos(a-\beta)-n\cos(a+n\beta).$$

Then, dividing by $2(1-\cos\beta)$, S, the sum of the required series would be obtained.

Note. Similarly the sum of the series

$$\sin \alpha + 2 \sin (\alpha + \beta) + 3 \sin (\alpha + 2\beta) + \dots + n \sin (\alpha + (n-1)\beta)$$
 would be obtained.

Examples XVII(b)

Sum the following series to n terms (Ex. 1 to 10):

1.
$$\sin a + \sin \left(a - \frac{\pi}{n}\right) + \sin \left(a - \frac{2\pi}{n}\right) + \cdots$$

2.
$$\cos a + \cos \left(a + \frac{2\pi}{n}\right) + \cos \left(a + \frac{4\pi}{n}\right) + \cdots$$

3.
$$\sin a - \sin 2a + \sin 3a - \cdots$$

4.
$$\cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \cdots$$

5.
$$\sin^3 a + \sin^3 3a + \sin^3 5a + \cdots$$

6.
$$\sin^2\theta - \sin^22\theta + \sin^23\theta - \sin^24\theta + \cdots$$

7.
$$\sin^4 a + \sin^4 2a + \sin^4 3a + \cdots$$

8.
$$\cos \theta - \sin 2\theta - \cos 3\theta + \sin 4\theta + \cos 5\theta - \sin 6\theta - \cdots$$

9.
$$\sin a \sin 2a + \sin 2a \sin 3a + \sin 3a \sin 4a + \cdots$$

10.
$$\cos a \cos 3a + \cos 3a \cos 5a + \cos 5a \cos 7a + \cdots$$

Find the sum of the following series (Ex. 11 to 14):

11.
$$\cos \frac{\pi}{19} + \cos \frac{3\pi}{19} + \cos \frac{5\pi}{19} + \dots + \cos \frac{17\pi}{19}$$

12.
$$\sin 5^{\circ} + \sin 77^{\circ} + \sin 149^{\circ} + \dots + \sin 293^{\circ}$$
.

13.
$$\sin \frac{2\pi}{n} + \sin \frac{4\pi}{n} + \sin \frac{6\pi}{n} + \dots + \sin \frac{2n\pi}{n}$$

14.
$$\sin na + \sin (n-1)a + \sin (n-2)a + \cdots$$
 to $2n$ terms.

15. Prove that

(i)
$$\frac{\sin \theta + \sin 3\theta + \sin 5\theta + \cdots \text{ to } n \text{ terms}}{\cos \theta + \cos 3\theta + \cos 5\theta + \cdots \text{ to } n \text{ terms}} = \tan n\theta.$$

(ii)
$$\sin^2 a + \sin^2 \left(a + \frac{2n}{n}\right) + \sin^2 \left(a + \frac{4n}{n}\right) + \cdots$$
 to n terms

Sum to n terms (Ex. 16 to 28):

- 16. sec a sec $2a + \sec 2a$ sec $3a + \sec 3a$ sec $4a + \cdots$
- 17. $\frac{1}{\sin \theta \sin 2\theta} + \frac{1}{\sin 2\theta \sin 3\theta} + \frac{1}{\sin 3\theta \sin 4\theta} + \cdots$
- 18. $\frac{1}{\cos a + \cos 3a} + \frac{1}{\cos a + \cos 5a} + \frac{1}{\cos a + \cos 7a} + \cdots$
- 19. $\cot \theta \cot 2\theta + \cot 2\theta \cot 3\theta + \cot 3\theta \cot 4\theta + \cdots$
- 20. $\tan a + 2 \tan 2a + 4 \tan 4a + 8 \tan 8a + \cdots$

[
$$tan a = cot a - 2 cot 2a$$
]

- 21. $\sin 2\theta \sin^2 \frac{2\theta}{2} + \sin 3\theta \sin^2 \frac{3\theta}{2} + \sin 4\theta \sin^2 \frac{4\theta}{2} + \cdots$
- 22. $\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 3^2x} + \frac{\sin 3^3x}{\cos 3^5x} + \cdots$

[1st term = $\frac{1}{2}$ (tan 3x - tan x)]

23.
$$\tan^{-1} \frac{1}{1+1+1^2} + \tan^{-1} \frac{1}{1+2+2^2} + \tan^{-1} \frac{1}{1+3+3^2} + \cdots$$

24.
$$\tan^{-1} \frac{2}{1+1.3} + \tan^{-1} \frac{2}{1+3.5} + \tan^{-1} \frac{2}{1+5.7} + \cdots$$

25.
$$\cot^{-1}(2.1^2) + \cot^{-1}(2.2^2) + \cot^{-1}(2.3^2) + \cdots$$

26.
$$\tan x + \frac{1}{2} \tan \frac{1}{2} x + \frac{1}{2^2} \tan \frac{1}{2^2} x + \cdots$$

- 27. $\cos x \cos 2x \cos 3x + \cos 2x \cos 3x \cos 4x + \cdots$
- 28. $\cos \theta + 2 \cos 2\theta + 3 \cos 3\theta + \cdots + n \cos n\theta$.
- 29. Find the sum of the series:
 - (i) $\sin a + \sin 2a + \sin 3a + \cdots$ to n terms
- and (ii) $\sin a + \sin 3a + \sin 5a + \cdots$ to n terms

and hence deduce respectively the sums of the series

- (a) $1+2+3+\cdots$ to n terms
- and (b) $1+3+5+\cdots$ to n terms.
 - 30. Sum the series

 $\tan x \tan 2x + \tan 2x \tan 3x + \cdots + \tan nx \tan (n+1)x$ and hence deduce the sum of the series

$$1.2 + 2.3 + \cdots + n(n+1)$$
.

- 31. If β be the exterior angle of a regular polygon of n sides, show that
 - (i) $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \cdots$ to n terms = 0.
 - (ii) $\cos a + \cos (a + \beta) + \cos (a + 2\beta) + \cdots$ to n terms = 0.
- 32. A regular polygon of n sides is inscribed in a circle of radius a; prove that
- (i) the sum of the lengths of the perpendiculars drawn from the angular points upon any diameter is zero;
- (ii) the sum of the lengths of the lines joining any vertex to each of the other vertices is $2a \cot \frac{\pi}{2n}$.

Sec. C-ELIMINATION

118. The elimination of trigonometrical functions from given equations is a very important and common mathematical problem. There are no set rules to effect the elimination. The form of the equations will often suggest special methods, and in addition to the usual algebraical artifices, we shall always have at our disposal the identical relations existing among the trigonometrical functions.

The following examples will illustrate some useful methods of elimination.

Ex. 1. Eliminate 0 from the equations

$$a \cos \theta + b \sin \theta + c = 0$$

$$a'\cos\theta + b'\sin\theta + c' = 0$$
.

From the given equations, we have by cross-multiplication,

$$\frac{\cos \theta}{bc'-b'c} = \frac{\sin \theta}{ca'-c'a} = \frac{1}{ab'-a'b}.$$

$$\therefore \cos \theta = \frac{bc' - b'c}{ab' - a'b}, \text{ and } \sin \theta = \frac{ca' - c'a}{ab' - a'b}.$$

Squaring and adding, we get

$$(bc'-b'c)^2+(ca'-c'a)^2=(ab'-a'b)^2$$

as the required eliminant.

Ex. 2. Eliminate 0 from the equations

$$x \sin \theta + y \cos \theta = 2a \sin 2\theta$$

$$x \cos \theta - y \sin \theta = a \cos 2\theta$$
.

Solving as simultaneous equations in x and y, we have

$$x = a(\cos 2\theta \cos \theta + 2 \sin 2\theta \sin \theta)$$

$$= a[\cos(2\theta - \theta) + \sin 2\theta \sin \theta]$$

$$=a(\cos\theta+2\sin^2\theta\cos\theta)$$
;

$$y = a(2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta)$$

$$= a(\sin \theta + \sin 2\theta \cos \theta) = a(\sin \theta + 2 \sin \theta \cos^2 \theta).$$

$$\therefore x + y = a(\sin \theta + \cos \theta)(1 + 2\sin \theta \cos \theta)$$

$$= a(\sin \theta + \cos \theta)(\sin \theta + \cos \theta)^2 = a(\cos \theta + \sin \theta)^3.$$

Similarly,

$$x - y = a(\cos \theta - \sin \theta)(1 - 2 \sin \theta \cos \theta)$$
$$= a(\cos \theta - \sin \theta)^{3}.$$

$$\therefore a^{\frac{1}{3}}(\cos\theta + \sin\theta) = (x+y)^{\frac{1}{3}} \qquad \cdots \qquad (i)$$

$$a^{\frac{1}{3}}(\cos\theta - \sin\theta) = (x - y)^{\frac{1}{3}}.$$
 ··· (ii)

Hence, squaring and adding (i) and (ii), we have,

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}},$$

as the required eliminant.

Ex. 3. Eliminate x and y from the equations $a \sin^2 x + b \cos^2 x = c$, $b \sin^2 y + a \cos^2 y = d$, $a \tan x = b \tan y$.

From the first equation, we have $a \sin^2 x + b \cos^2 x = c(\sin^2 x + \cos^2 x)$.

$$\therefore (a-c)\sin^2 x = (c-b)\cos^2 x.$$

$$\therefore \tan^2 x = \frac{c-b}{a-c}$$

From the second equation, we have similarly $b \sin^2 y + a \cos^2 y = d(\sin^2 y + \cos^2 y)$.

$$\therefore \tan^2 y = \frac{d-a}{b-d}.$$

From the third equation,

$$a^2 \tan^2 x = b^2 \tan^2 y.$$

$$\therefore \frac{a^2(c-b)}{a-c} = \frac{b^2(d-a)}{b-d}.$$

This, when simplified, reduces to

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}$$
, the required eliminant.

Examples XVII(c)

Eliminate θ from the following pair of equations (Ex. 1 to 15):

- 1. $\cot \theta (1 + \sin \theta) = 4a$. $\cot \theta (1 - \sin \theta) = 4b$.
- 2. $x = a \cos \theta + b \cos 2\theta$. $y = a \sin \theta + b \sin 2\theta$.
- 3. $x = \tan \theta + \tan 2\theta$. $y = \cot \theta + \cot 2\theta$.
- 4. $a \sin \theta + b \cos \theta = 1$. $a \csc \theta - b \sec \theta = 1$.
- 5. $x = \sin \theta + \cos \theta \sin 2\theta$ $y = \cos \theta + \sin \theta \sin 2\theta$.
- 6. $x + a = a(2 \cos \theta \cos 2\theta)$ $y = a(2 \sin \theta - \sin 2\theta)$.
- 7. $x=3 \sin \theta \sin 3\theta$ $y=\cos 3\theta + 3 \cos \theta$.
- 8. $x \cot \theta + \tan \theta$ $y = \sec \theta - \cos \theta$.
- 9. $x \sin \theta y \cos \theta = \sqrt{x^2 + y^2}$ $\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} = \frac{1}{x^2 + y^2}.$
- $10. \quad \frac{x}{a} = \cos \theta + \cos 2\theta$
 - $\frac{y}{b} = \sin \theta + \sin 2\theta.$

11.
$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

$$\frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0.$$

12.
$$\frac{x}{a} \cos \theta - \frac{y}{b} \sin \theta = \cos 2\theta$$

$$\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 2\sin 2\theta.$$

13.
$$x = \csc \theta - \sin \theta$$

 $y = \sec \theta - \cos \theta$.

14.
$$\sin \theta + \cos \theta = a$$

 $\sin^3 \theta + \cos^3 \theta = b$.

15.
$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$
$$x\sin\theta - y\cos\theta = (a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{1}{2}}.$$

Eliminate θ and ϕ from the following equations (Ex. 16 to 19):—

- 16. $\sin \theta + \sin \phi = x$, $\cos \theta + \cos \phi = y$, $\theta \phi = a$.
- 17. $\tan \theta + \tan \phi = a$, $\cot \theta + \cot \phi = b$, $\theta + \phi = a$.
- 18. $a \sin^2 \theta + b \cos^2 \theta = a \cos^2 \phi + b \sin^2 \phi = 1$,

 $a \tan \theta = b \tan \phi$.

19.
$$\sin \theta + \sin \phi = a$$
, $\cos \theta + \cos \phi = b$, $\sin 2\theta + \sin 2\phi = 2c$.

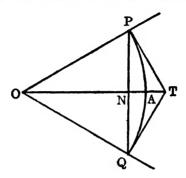
20. If
$$(a+b)$$
 tan $(\theta-\phi)=(a-b)$ tan $(\theta+\phi)$ and $a\cos 2\phi + b\cos 2\theta = c$, show that
$$a^2 - b^2 + c^2 = 2ac\cos 2\phi.$$

APPENDIX

1. To prove that

$$\sin \theta < \theta < \tan \theta$$
,

where 0 is the circular measure of any positive acute angle.



Let AOP be a positive acute angle whose radian measure is θ , and let QOA be an equal angle on the other side of OA. With centre O and any radius, a circle is drawn cutting OP, OA, OQ at P, A, Q respectively. PQ is joined cutting OA at N. The triangles OPN and OQN are easily seen to be congruent, so that PN = QN and PNQ is perpendicular to OA. The tangent PT to the circle at P cuts OA at T, therefore $\angle OPT$ is a right angle. TQ being joined, the triangles OPT and OQT are easily proved to be congruent, so that TP = TQ.

The figure is thus symmetrical about OA.

Then, from the figure,

$$\sin \theta = \frac{PN}{OP} = \frac{1}{2} \cdot \frac{PQ}{OP}$$

$$\theta = \frac{\text{arc } PA}{OP} = \frac{1}{2} \frac{\text{arc } PAQ}{OP}$$

$$\tan \theta = \frac{PT}{OP} = \frac{1}{2} \frac{PT + QT}{OP}.$$

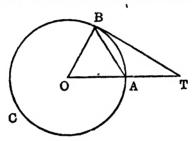
Now, we may take it as axiomatic that the straight line PQ is less than the curved arc PAQ, and that the curved arc PAQ which always bends the same way, being within the triangle PTQ, is less than PT+TQ.

Hence, since PQ < arc PAQ < PT + QT,

we have, on dividing throughout by 2OP, $\sin \theta < \theta < \tan \theta$.

Alternative method :

Let ABC be a circle whose centre is O and radius r,



Let $AOB = \theta$ radians.

Draw BT the tangent at B to meet OA produced at T; then $BT = r \tan \theta$.

We know that if the angle of a sector of a circle of radius r is θ radians, its area = $\frac{1}{2}r^2\theta$.

Now, from the figure it is obvious that

$$\triangle OAB < sector OAB < \triangle OBT$$
.

 $\therefore \frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2\theta < \frac{1}{2}r.r \tan \theta,$

i.e., $\sin \theta < \theta < \tan \theta$.

Cor. If now θ becomes infinitely small, we can prove

Lt
$$\frac{\sin \theta}{\theta} = 1$$
,

$$\underset{\theta \to 0}{\text{Lt}} \cos \theta = 1,$$

and Lt
$$\frac{\tan \theta}{\theta} = 1$$
.

For, since, $\sin \theta < \theta < \tan \theta$, we get, by dividing by $\sin \theta$,

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

This is true, however small θ may be, provided it is positive. When θ becomes infinitely small, OP and ON practically come into coincidence, so that

 $\cos \theta = \frac{ON}{OP}$ ultimately becomes 1.

Hence, $\int_{\theta \to 0} t \cos \theta = 1$.

In that case $\frac{1}{\cos \theta}$ also tends to the value 1. But $\frac{\theta}{\sin \theta}$ always lies between 1 and $\frac{1}{\cos \theta}$ which ultimately come into coincidence, and so $\frac{\theta}{\sin \theta}$ also ultimately coincides with 1.

Thus, $\frac{\sin \theta}{\theta} = 1$ in the limit.

Again, from

 $\sin \theta < \theta < \tan \theta$,

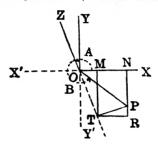
we get by dividing by $\tan \theta$,

$$\cos\theta<\frac{\theta}{\tan\theta}<1,$$

and as $\theta \to 0$, $\cos \theta \to 1$ and $\frac{\theta}{\tan \theta}$ always lying between $\cos \theta$ and 1 which come into coincidence, $\frac{\theta}{\tan \theta} = 1$ in the limit, and so $Lt \frac{\tan \theta}{\theta \to 0} = 1$.

Hence, the results.

2. Formulæ for $\sin (A+B)$ and $\cos (A+B)$ where A and B are of any magnitude. (Generalization of Art. 33)



In Article 33, formulæ for $\sin (A+B)$ and $\cos (A+B)$ were deduced geometrically with a figure in which A and B were acute and (A+B) less than 90°. We now prove them in a more general case.

Let a revolving line, starting from OX, trace out an angle XOZ = A and further trace out an angle ZOP = B, so that the total angle traced out is (A + B). From any point P on the final position of the revolving line, PN and PT are drawn perpendiculars to OX and OZ (produced if necessary, as in the above figure), and from T perpendiculars TM and TR are drawn on OX and PN (produced if necessary).

In the figure above, $\angle POT = B - 180^{\circ}$, and since PN and PT are perpendiculars to OX and OZ respectively, $\angle TPR = \angle TON = 180^{\circ} - \angle XOZ$, i.e., $180^{\circ} - A$.

In considering $\sin (A+B)$ and $\cos (A+B)$ from the triangle NOP, it is to be noted that PN is negative and ON and OP are positive.

I. we consider only the positive magnitudes of the sides of the acute-angled triangles OTM, PTR and OPT, then FN with its proper sign may be written as -(TM-PR), and ON with its proper sign may be written as OM+TR.

Now, from the figure,

$$\sin (A + B) = \frac{PN}{OP} = -\frac{TM - PR}{OP}$$

Again,

$$\cos (A+B) = \frac{ON}{OP} = \frac{OM + RT}{OP}$$

$$= \frac{OM}{OT} \frac{OT}{OP} + \frac{RT}{P} \frac{PT}{OP}$$

$$= \cos TOM \cos POT + \sin TPR \sin POT$$

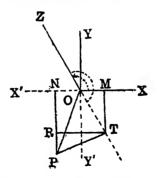
$$= \cos (180^{\circ} - A) \cos (B - 180^{\circ})$$

$$+ \sin (180^{\circ} - A) \sin (B - 180^{\circ})$$

$$= (-\cos A)(-\cos B) + \sin A (-\sin B)$$

$$= \cos A \cos B - \sin A \sin B.$$

3. Formulæ for sin (A - B) and cos (A - B) in a more general case. (Generalization of Art. 34)



Here, XOZ measured counter-clockwise is A and ZOP measured clockwise has magnitude B so that XOP measured

clockwise is A-B. From P, PN and PT are drawn perpendiculars on OX and OZ (produced in this figure), and from T, TM and TR are drawn perpendiculars on OX and PN.

In the present figure, magnitudes of the acute angles TOM and POT are $180^{\circ} - A$ and $B - 180^{\circ}$ respectively, and noting that PNOT is a cyclic quadrilateral (\angle ⁵ N and T being right angles), $\angle RPT = \angle TOM = 180^{\circ} - A$ in magnitude.

Now, we see that in considering $\sin (A-B)$ and $\cos (A-B)$ from the triangle NOP, PN and ON are of negative signs.

Hence,

$$\sin (A - B) = \frac{PN}{OP}$$

$$= -\frac{MT + PR}{OP},$$

[where the magnitudes of MT, PR, etc. only are considered]

$$= -\frac{MT}{OT} \cdot \frac{OT}{OP} - \frac{PR}{PT} \cdot \frac{PT}{OP}$$

$$= -\sin TOM \cos POT - \cos RPT \sin POT$$

$$= -\sin (180^{\circ} - A) \cos (B - 180^{\circ})$$

$$-\cos (180^{\circ} - A) \sin (B - 180^{\circ})$$

$$= -\sin A (-\cos B) - (-\cos A)(-\sin B)$$

$$= \sin A \cos B - \cos A \sin B.$$

Similarly,

$$\cos (A - B) = \frac{ON}{OP} \qquad [\text{ where } ON \text{ is taken with proper sign }]$$

$$= -\frac{RT - OM}{OP} \qquad [\text{ where magnitudes only of } \\ RT, OM \text{ etc. are considered }]$$

$$= -\frac{RT}{PT} \cdot \frac{PT}{OP} + \frac{OM}{OT} \cdot \frac{OT}{OP}$$

$$= -\sin^* RPT \sin POT + \cos TOM \cos POT$$

$$= -\sin (180^\circ - A) \sin (B - 180^\circ)$$

$$+ \cos (180^\circ - A) \cos (B - 180^\circ)$$

=
$$-\sin A (-\sin B) + (-\cos A)(-\cos B)$$

= $\cos A \cos B + \sin A \sin B$.

4. A few particular cases of $\sin (A \pm B)$, $\cos (A \pm B)$.

Case I. In the case A and B are both acute and $(A+B) > 90^{\circ}$.

Construction is 'same as in Art. 33. Here, Q, the foot of the perpendicular will fall on XO produced.

$$\angle TPR = 90^{\circ} - \angle TRP - \angle TRO = \angle ROS = \angle A.$$

$$\sin (A + B) = \sin XOP$$

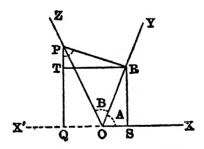
$$= \frac{PQ}{OP} = \frac{QT + TP}{OP}$$

$$= \frac{RS + PT}{OP} - \frac{RS}{OP} + \frac{PT}{OP}$$

$$= \frac{RS}{OR} \cdot \frac{OR}{OP} + \frac{PT}{PR} \cdot \frac{PR}{OP}$$

$$= \sin A \cos B + \cos TPR \sin B$$

 $= \sin A \cos B + \cos A \sin B.$ $= \sin A \cos B + \cos A \sin B.$



$$\cos (A + B) = \cos XOP = -\frac{OQ}{OP} \qquad \begin{array}{c} [\text{Magnitude of } OQ \text{ being considered }] \\ = -\frac{SQ - SO}{OP} = \frac{OS}{OP} - \frac{SQ}{OP} = \frac{OS}{OP} - \frac{TR}{OP} \end{array}$$

$$= \frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP}$$

$$= \cos A \cos B - \sin TPR \sin B$$

$$= \cos A \cos B - \sin A \sin B.$$

Case II. In the case A is obtuse and B is acute and $(A+B) < 180^{\circ}$.

Construction is same as in Art. 33.

Here,
$$\angle TPR = 180^{\circ} - \angle RPQ = \angle ROQ = 180^{\circ} - A$$
.

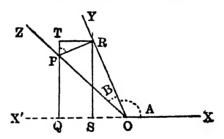
$$\sin TPR = \sin A ; \cos TPR = -\cos A.$$

$$\sin (A+B) = \sin XOP = \frac{PQ}{OP} = \frac{QT-PT}{OP} = \frac{RS-PT}{OP}$$

$$= \frac{RS}{OP} - \frac{PT}{OP} = \frac{RS}{OR} \cdot \frac{OR}{OP} - \frac{PT}{PR} \cdot \frac{PR}{OP}$$

$$= \sin A \cos B - \cos TPR \sin B.$$

$$= \sin A \cos B + \cos A \sin B.$$



$$\cos (A+B) = \cos XOP = -\frac{OQ}{OP} \qquad [\text{Magnitude of } OQ \text{ being considered }]$$

$$= -\frac{OS + SQ}{OP} = -\frac{OS}{OP} - \frac{SQ}{OP}$$

$$= -\frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP}$$

$$= \cos A \cos B - \sin TPR \sin B$$

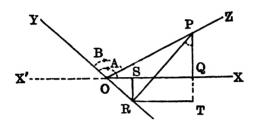
$$= \cos A \cos B - \sin A \sin B.$$

Case III. In the case A and B are both obtuse and (A-B) is acute.

Construction is same as in Art. 34.

Here, $\angle TPR = \angle ROS = 180^{\circ} - A$.

 $\sin (A - B) = \sin POQ$



$$= \frac{PQ}{OP} = \frac{PT - RS}{OP}$$

$$= \frac{PT}{OP} - \frac{RS}{OP} = \frac{PT}{PR} \cdot \frac{PR}{OP} - \frac{RS}{OR} \cdot \frac{OR}{OP}$$

$$= \cos TPR \sin POR - \sin ROS \cos POR$$

$$= \cos (180^{\circ} - A) \sin (180^{\circ} - B).$$

$$- \sin (180^{\circ} - A) \cos (180^{\circ} - B)$$

$$= - \cos A \sin B - \sin A (- \cos B)$$

$$= \sin A \cos B - \cos A \sin B.$$

$$\cos (A - B) = \cos POQ$$

$$= \frac{OQ}{OP} = \frac{OS + SQ}{OP} = \frac{OS + RT}{OP} = \frac{OS}{OP} + \frac{RT}{OP}$$

$$= \frac{OS}{OR} \cdot \frac{OR}{OP} + \frac{RT}{PR} \cdot \frac{PR}{OT}$$

$$= \cos ROS \cdot \cos POR + \sin TPR \cdot \sin POR$$

$$= \cos (180^{\circ} - A) \cos (180^{\circ} - B)$$

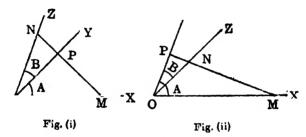
$$+ \sin (180^{\circ} - A) \sin (180^{\circ} - B)$$

$$= (-\cos A)(-\cos B) + \sin A \sin B$$

$$= \cos A \cos B + \sin A \sin B.$$

Note. Other particular cases of the above four formulæ can easily be proved exactly in the same way by drawing the corresponding figures in each case and making the same construction as in Arts. 33 and 34 for (A+B) and (A-B) respectively.

5. An alternative Method of proof for $\sin (A \pm B)$, $\cos (A \pm B)$. [See Arts. 33, 34]



Let $\angle XOY = A$; $\angle YOZ = B$; in Fig. (i), $\angle XOZ = A + B$ (< 90°); in Fig. (ii), $\angle XOZ = A - B$ (A > B) [A and B being positive and acute].

Through any point P on OY, the common arm of two angles, draw a straight line MN perpendicular to OY, meeting OX in M and OZ in N.

Then, $\triangle MON = \triangle MOP + \triangle NOP$.

 $\therefore \frac{1}{2}OM.ON \sin (A \pm B) = \frac{1}{2}OM.OP \sin A \pm \frac{1}{2}ON.OP \sin B$ [Art. 88(i)]

$$\therefore \sin (A \pm B) = \sin A \cdot \frac{OP}{ON} \pm \frac{OP}{OM} \sin B$$

$$= \sin A \cos B \pm \cos A \sin B.$$

$$\cos (A \pm B) = \cos MON = \frac{OM^2 + ON^2 - MN^2}{2OM \cdot ON}$$
[Art. 83]

$$\frac{(OP^2 + PM^2) + (OP^2 + PN^2) - (MP \mp PN)^2}{2OM \ ON}$$

$$= \frac{OP^2 \mp MP.PN}{OM.ON} = \frac{OP.OP}{OM.ON} \mp \frac{MP.PT}{OM.ON}$$

$$= \cos A \cos B \mp \sin A \sin B.$$

6. Geometrical proof of the expansion of tan (A + B).

The figure and the construction are the same as in Art. 33.

$$\tan (A+B) = \frac{PQ}{OQ} = \frac{RS + PT}{OS - TR}$$

$$= \frac{\frac{RS}{OS} + \frac{PT}{OS}}{1 - \frac{TR}{OS}} = \frac{\frac{RS}{OS} + \frac{PT}{OS}}{1 - \frac{TR}{TP} \cdot \frac{TP}{OS}}$$

Now,
$$\frac{RS}{OS} = \tan A$$
 and $\frac{TR}{TP} = \tan TPR = \tan A$.

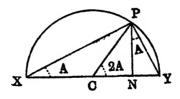
The triangles ROS, TPR are similar.

$$\therefore \frac{TP}{OS} = \frac{PR}{OR} = \tan B.$$

$$\therefore \tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Note. Similarly the expansion of $\tan (A - B)$ can be proved geometrically from the figure and construction of Art. 34.

7. Geometrical proof of the formulæ for 2A.



Let XPY be a semi-circle, XY the diameter and C the centre.

Draw PN perpendicular to XY.

Let
$$\angle PXY = A$$
; then $\angle PCY = 2A$.
 $\angle NPY = 90^{\circ} - \angle PYN = \angle PXY = A$.
 $\sin 2A = \frac{PN}{CP} = \frac{2PN}{2CP} = \frac{2PN}{XY} = 2\frac{PN}{XP} \cdot \frac{XP}{XY}$

$$= 2 \sin PXN \cdot \cos PXY = 2 \sin A \cos A.$$
 $\cos 2A = \frac{CN}{PC} = \frac{2CN}{2CP} = \frac{2CN}{XY} = \frac{CN + CN}{XY}$

$$= \frac{(XN - XC) + (CY - YN)}{XY} = \frac{XN \cdot XP}{XY} = \frac{YN}{XY} \cdot \frac{PY}{YY} \cdot \frac{YN}{XY}$$

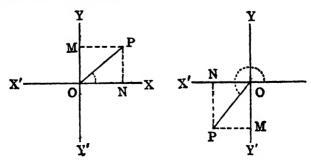
$$= \cos A \cdot \cos A - \sin A \cdot \sin A$$

$$= \cos^2 A - \sin^2 A.$$

$$\tan 2A = \frac{PN}{CN} = \frac{2PN}{2CN} = \frac{2PN}{XN - YN}$$

$$\frac{2PN}{XN} = \frac{2PN}{XN} = \frac{2PN}{XN} \cdot \frac{2PN}{XN} \cdot \frac{2PN}{XN} \cdot \frac{2PN}{XN} \cdot \frac{2PN}{NN} \cdot \frac{2PN}{$$

8. Trigonometrical ratios of generalized angle defined by projections.



Let XOX' and YOY' be a pair of rectangular axes intersecting at the point O and let an angle θ , of any magnitude (positive or negative) be generated by the revolution of OP from its initial position OX to its present position. Then the trigonometrical ratios of the generalized angle θ are defined as follows:

$$\sin \theta = \frac{\text{projection of } OP \text{ on } y\text{-axis}}{OP}$$

$$\cos \theta = \frac{\text{projection of } OP \text{ on } x\text{-axis}}{OP}$$

$$\tan \theta = \frac{\text{projection of } OP \text{ on } y\text{-axis}}{\text{projection of } OP \text{ on } x\text{-axis}}$$

$$\cos \theta = \frac{OP}{\text{projection of } OP \text{ on } y\text{-axis}}$$

$$\sec \theta = \frac{OP}{\text{projection of } OP \text{ on } x\text{-axis}}$$

$$\cot \theta = \frac{OP}{\text{projection of } OP \text{ on } x\text{-axis}}$$

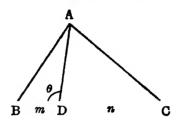
$$\cot \theta = \frac{OP}{\text{projection of } OP \text{ on } x\text{-axis}}$$

In the above definitions, projection means algebraic projection; that is, we should consider not only the magnitude but also the sign of the projection; and with the usual convention the projection would be considered positive if they are along OX and OY and considered negative if they are along OX' and OY'. By convention, OP is always considered positive. From these definitions, the signs of the trigonometrical ratios can be easily determined according to the position of OP in one or other of the four quadrants. In the above figures, the position of OP in two quadrants only (1st and 3rd) are shown.

Note 1. From the above definitions, it is clear that if OX be a fixed line, and if l be the length of any straight line OP inclined at an angle θ to OX, then the projection of OP along OX is l cos θ whatever be the magnitude of the angle θ .

Note 2. The Addition and Subtraction Theorem for generalized angles can also be proved by the method of projection.

9. Two important trigonometrical relations.



If D be any point in the base BC of a triangle ABC, and if AD divides BC into two parts m and n (BD = m, CD = n) and the angle A into two parts a and $\beta (\angle BAD = a, \angle CAD = \beta)$ and if the angle ADB be θ , then

- (i) $(m+n) \cot \theta = n \cot \beta m \cot \alpha$.
- (ii) $(m+n) \cot \theta = m \cot C n \cot B$.

We have

$$\frac{m}{n} = \frac{BD}{DC} = \frac{BD}{AD} \frac{AD}{DC} = \sin \frac{BAD}{\sin \frac{ACD}{DAC}}$$

$$= \frac{\sin \alpha}{\sin (\theta + \alpha)} = \sin \frac{(\theta - \beta)}{\sin \beta} \left[\begin{array}{c} \therefore \angle ABD = \pi - (\alpha + \theta). \\ \angle ACD = \theta - \beta. \end{array} \right]$$

$$= \frac{\sin \alpha}{\sin \beta} \left(\sin \theta \cos \beta - \cos \theta \sin \beta \right)$$

$$\sin \beta \left(\sin \theta \cos \alpha + \cos \theta \sin \alpha \right)$$

Dividing the numerator and the denominator by $\sin a \sin b \sin \theta$, we have

$$\frac{m}{n} = \frac{\cot \beta - \cot \theta}{\cot \alpha + \cot \theta}$$

$$\therefore (m+n) \cot \theta = n \cot \beta - m \cot \alpha.$$

Again,

$$\frac{m}{n} = \frac{\sin BAD \sin ACD}{\sin ABD \sin DAC}$$

$$= \frac{\sin (\theta + B)}{\sin B} \frac{\sin C}{\sin (\theta - C)} \begin{bmatrix} \because \angle BAD = \pi - (\theta + B), \\ \angle DAC = \theta - C \end{bmatrix}$$

$$= \frac{\sin C (\sin \theta \cos B + \cos \theta \sin B)}{\sin B (\sin \theta \cos C - \cos \theta \sin C)}$$

Dividing the numerator and the denominator by $\sin B \sin C \sin \theta$, we have

$$\frac{m}{n} = \cot B + \cot \theta.$$

$$\cot C - \cot \theta.$$

$$\therefore (m+n) \cot \theta = m \cot C - n \cot B.$$

10. Note on Art. 90.

Let us denote the formulæ of Arts. 82, 83, 84 by (I), (II), (III). We have seen in Art. 90, that (II) can be deduced from (III). We shall now show how any one of these can be deduced from any other of the rest.

To deduce (I) from (III).

Substituting the value of b from the second relation of Art. 84 in the first,

$$a = (c \cos A + a \cos C) \cos C + c \cos B.$$

$$\therefore a (1 - \cos^2 C) = c (\cos A \cos C + \cos B)$$

$$= c \left\{ \cos A \cos C - \cos (A+C) \right\}$$

$$\left\{ \therefore A+B+C=\pi \right\}$$

$$\therefore a \sin^2 C = c \sin A \sin C. \quad \therefore a/\sin A = c/\sin C.$$

Similarly, substituting the value of c in the first relation, we get

 $a/\sin A = b/\sin B$. Hence, etc.

To deduce (II) and (III) from (I).

(i) Putting each of the ratios of Art. 82 equal to k, we get

$$a = k \cdot \sin A \; ; \; b = k \cdot \sin B \; ; \; c = k \cdot \sin C.$$

$$b^{2} + c^{2} - a^{2} = k^{2} \left(\sin^{2}B + \sin^{2}C - \sin^{2}A \right)$$

$$k^{2} \cdot 2 \sin B \sin C$$

$$= \sin^{2}B + \sin (C + A) \sin (C - A)$$

$$2 \sin B \sin C$$

$$= \frac{\sin B \left\{ \sin B + \sin (C - A) \right\}}{2 \sin B \sin C}$$

$$[\because \sin (C + A) = \sin (\pi - B) = \sin B \right]$$

$$= \sin B \left\{ \sin (C + A) + \sin (C - A) \right\}$$

$$= 2 \sin B \sin C \cos A$$

$$= 2 \sin B \sin C \cos A$$

$$= 2 \sin B \sin C \cos A$$

(ii)
$$b \cos C + c \cos B = k (\sin B \cos C + \sin C \cos B)$$

= $k \sin (B + C) = k \sin A$
= a . [: $A + B + C = \pi$]

To deduce (I) and (III) from (II).

(i)
$$\sin^2 A = 1 - \cos^2 A$$

$$= 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 = \frac{4b^2c^3 - (b^2 + c^2 - a^2)^2}{4b^2c^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4b^2c^2}$$

$$= \frac{(a + b + c)(b + c - a)(c + a - b)(a + b - c)}{4b^2c^2}$$

$$= \frac{K}{4b^2c^2} \text{ Say}.$$

$$\therefore \frac{\sin^2 A}{a^2} = \frac{K}{4a^2b^2c^2};$$

similarly, $\frac{\sin^2 B}{b^2}$ and $\frac{\sin^2 C}{c^2}$ each $=\frac{K}{4a^2b^2c^2}$. $\therefore \frac{\sin^2 A}{c^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2}$; hence, etc.

(ii) Adding 2nd and 3rd relations of the formulæ of Art. 83, we get

$$b^2 + c^2 = b^2 + c^2 + 2a^2 - 2ca \cos B - 2ab \cos C$$

Now, transposing and dividing by 2a, we get $a = b \cos C + c \cos B$; etc.

Miscellaneous Examples III

- 1. The angles of a triangle are as 4:5:6. Express them in circular measure.
- 2. The angles of a triangle are in A.P. and the greatest is double the least; express the angles in degrees, and in radians.
- 3. The number of degrees in one of the acute angles of a right-angled triangle is three-tenths of the number of grades in the other: determine the angles in degrees.
- 4. Compare the areas of two circles in which the circumference of one is equal to an arc of 60° of the other.
- 5. A railway train is travelling on a curve of half-a-mile radius at the rate of 20 miles an hour; through what angle has it turned in 10 seconds?
- 6. An arc of a circle whose radius is 7 inches, subtends an angle of 15° 39′ 7″; what angle will an arc of the same length subtend in a circle whose radius is 2 inches?

Prove the following identities (Ex. 7 to 22):—

- 7. $\sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta = \tan \theta + \cos \theta$.
- 8. $\sin^2\theta (1 + \cot^2\theta) + \cos^2\theta (1 + \tan^2\theta) = 2$.

9.
$$(\tan \theta + \sec \theta)^2 = \frac{1 + \sin \theta}{1 - \sin \theta}$$
 [C. U. 1934]

- 10. $2(\sin^6\theta + \cos^6\theta) 3(\sin^4\theta + \cos^4\theta) + 1 = 0$.
- 11. $\frac{\tan x \cot y}{\tan y \cot x} = \tan x \cot y.$
- 12. $(\sin x \cos y + \cos x \sin y)^2 + (\cos x \cos y \sin x \sin y)^2 = 1$.

13.
$$\sin^4 x + \cos^4 x = 1 - 2 \sin^2 x \cos^2 x$$
.

14.
$$\sin^8 x - \cos^8 x = (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x \cos^2 x)$$
.

15.
$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$$
.

16.
$$(1 + \sec \theta + \tan \theta)(1 - \sec \theta + \tan \theta) = 2 \tan \theta$$
.

17.
$$\frac{\cos \theta}{1 - \tan \theta} + \frac{\sin \theta}{1 - \cot \theta} = \sin \theta + \cos \theta.$$

18.
$$(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$$
.

19.
$$\cot^2 x \cdot \frac{\sec x - 1}{1 + \sin x} + \sec^2 x \cdot \frac{\sin x - 1}{1 + \sec x} = 0$$
.

20.
$$(\sin x + \cos x)(\tan x + \cot x) = \sec x + \csc x$$
.

21.
$$(\sin \theta + \sec \theta)^2 + (\cos \theta + \csc \theta)^2 = (1 + \sec \theta \csc \theta)^2$$
.

22.
$$\frac{1-\sin\theta\cos\theta}{\cos\theta(\sec\theta-\csc\theta)\sin^2\theta-\cos^2\theta}=\sin\theta.$$

23. If
$$a \cos^2 x + b \sin^2 x = c$$
, show that $\tan x = \pm \sqrt{\frac{c-a}{b-c}}$

24. If cosec
$$A + \text{cosec } B + \text{cosec } C = 0$$
, show that $(\Sigma \sin A)^2 = \Sigma \sin^2 A$.

25. If
$$x = a \cos \theta + b \sin \theta$$
 and $y = a \sin \theta - b \cos \theta$,
show that $x^2 + y^2 = a^2 + b^2$.

26. Express
$$\frac{\sin x}{\cos^3 x} + \frac{\cos x}{\sin^3 x}$$
 in terms of t,

where t stands for tan x.

- 27. If $\sin A = \frac{1}{2}$ and $\tan B = \sqrt{3}$, find the value of $\sin A \cos B + \cos A \sin B$.
- 28. If $\cos \theta = \frac{4}{3}$, find the value of $\frac{1 \tan^2 \theta}{1 + \tan^2 \theta}$

29. If 5 tan
$$\theta = 4$$
, find the value of
$$\frac{5 \sin \theta - 3 \cos \theta}{\sin \theta + 2 \cos \theta}$$

30. If
$$\frac{\sin x}{\sin y} = \sqrt{2}$$
, $\frac{\tan x}{\tan y} = \sqrt{3}$,

find x and y (given that x and y are acute angles).

31. Which of the statements is possible and which impossible, x, y and z being unequal real quantities?

(i) cosec
$$\theta = \frac{x^2 + y^2}{2xy}$$
 (ii) sec $\theta = \frac{x^2 - y^2}{x^2 + y^2}$

(iii)
$$\sin \theta = \frac{x^2 + y + z^2}{yz + zx + xy}$$
 (iv) $\tan \theta = \frac{2xy}{x^2 + y^2}$

- 32. Eliminate θ from the equations:
 - (i) $\sin \theta + \cos \theta = a$, $\sin \theta \cos \theta = b$.
 - (ii) $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \frac{x}{a}\sin\theta \frac{y}{b}\cos\theta = 1.$
 - (iii) $x = a \cos^3 \theta$, $y = b \sin^3 \theta$.
- 33. If $k \tan \theta = \tan k\theta$, prove that $\frac{\sin^2 k\theta}{\sin^2 \theta} = \frac{k^2}{1 + (k^2 1)\sin^2 \theta}$
- 34. If $\sec x \sec y + \tan x \tan y = \sec z$, then, $\sec x \tan y + \tan x \sec y = \pm \tan z$.
- 35. Show that $\left(\frac{1+\cot 60^{\circ}}{1-\cot 60^{\circ}}\right)^{3} = \frac{1+\cos 30^{\circ}}{1-\cos 30^{\circ}}$.
- 36. If $\tan x = \frac{\sin \theta \cos \theta}{\sin \theta + \cos \theta}$, prove that $\sin x = \frac{1}{\sqrt{2}}(\sin \theta \cos \theta)$.
- 37. Show that the product of $\sin x (1 + \sin x) + \cos x (1 + \cos x)$ and $\sin x (1 \sin x) + \cos x (1 \cos x)$ is equal to $2 \sin x \cos x$.
- 38. Find the height of a chimney when it is found that on walking towards it 250 feet, in a horizontal line through its base, the angular elevation changes from 45° to 75°.

- 39. The length of a kite string is 250 yards, and the angle of elevation of the kite is 30°. Find the height of the kite.
- 40. The angle of elevation of the top of a temple at a distance 300 feet is 30°; find its height.
- 41. Find the angle of elevation of the sun when the shadow of a pole 60 feet high, is $20 \sqrt{3}$ yards long.
- 42. The angles of elevation of a tower at two places due east of it and 200 feet apart are 45° and 30°; find the height of the tower.
- 43. An aeroplane leaves the earth at the point X and rises at a uniform speed. At the end of 15 seconds, an observer stationed at a distance of 660 feet from X, find the angular elevation of the aeroplane to be 45° ; at what rate in miles per hour is the aeroplane rising?
- 44. A ladder 45 feet long just reaches the top of a wall. The ladder makes an angle of 60° with the wall. Find the height of the wall and the distance of the foot of the ladder from the wall.
 - 45. If $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$, find the values of $\sin (A + B)$ and $\cos (A B)$.
 - 46. If $\tan A = \frac{5}{13}$ and $\tan B = \frac{9}{40}$, find the values of $\sin (A B)$ and $\cos (A B)$.
 - 47. If $\tan A = \frac{m+n}{m-n}$, and $\tan B = \frac{m-n}{m+n}$, find $\tan (A-B)$.
 - 48. If $\tan (x+y) = \frac{3}{6}$ and $\tan x = \frac{5}{6}$, find $\tan y$.
 - 49. If $\cos \theta = \frac{3}{8}$, find $\sin 2\theta$, $\tan 2\theta$, $\cos \frac{\theta}{2}$.
- 50. If $\cos x = \frac{1}{2}$, $\cos y = \frac{3}{2}(x \text{ and } y \text{ being positive acute angles})$, find the value of $\cos \frac{1}{2}(x-y)$.

- 51. If $\sin A = \frac{1}{\sqrt{2}}$, $\sin B = \frac{1}{\sqrt{3}}$, find the value of $\tan \frac{1}{2} (A+B) \cot \frac{1}{2} (A-B)$.
- 52. If $\sec x = \frac{17}{8}$, $\csc y = \frac{5}{4}$, find $\sec (x + y)$.
- 53. Prove that

$$\frac{2\cos 8\theta + 1}{2\cos \theta + 1} = (2\cos \theta - 1)(2\cos 2\theta - 1)(2\cos 4\theta - 1).$$

- 54. Show that $a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \cos (\theta a)$, if $\tan a = b/a$.
- 55. If $\sin^4 x + \cos^4 x = 1$, prove that x is zero or a multiple of $\frac{1}{2}\pi$.
- 56. If $\sqrt{2} \cos A = \cos B + \cos^3 B$, and $\sqrt{2} \sin A = \sin B \sin^3 B$, then $\sin (A B) = \pm \frac{1}{3}$.
 - 57. Prove that $\cos^2(a-\beta)-\sin^2(a+\beta)=\cos 2a\cos 2\beta$.
 - 58. Show that $\sin 18^{\circ} + \cos 18^{\circ} = \sqrt{2} \cos 27^{\circ}$.
 - 59. Show that whatever be the value of θ ,

 $\sin^2(\theta+a) + \sin^2(\theta+\beta) - 2\cos(\alpha-\beta)\sin(\theta+a)\sin(\theta+\beta)$ is independent of θ .

60. Show that

(i)
$$\frac{\sin \alpha}{\sin (\alpha - \beta) \sin (\alpha - \gamma)} + \frac{\sin \beta}{\sin (\beta - \gamma) \sin (\beta - \alpha)} + \frac{\sin \gamma}{\sin (\gamma - \alpha) \sin (\gamma - \beta)} = 0.$$

(ii)
$$\tan (\beta + \gamma - 2\alpha) + \tan (\gamma + \alpha - 2\beta) + \tan (\alpha + \beta - 2\gamma)$$

= $\tan (\beta + \gamma - 2\alpha) \tan (\gamma + \alpha - 2\beta) \tan (\alpha + \beta - 2\gamma)$.

- 61. If $\tan \frac{1}{2}\theta = \tan^3 \frac{1}{2}\phi$ and $\tan \phi = 2 \tan \alpha$, show that $\theta + \phi = 2\alpha$.
- 62. (i) If $\tan^2 x = 2 \tan^2 y + 1$, then $\cos 2x + \sin^2 y = 0$.
 - (ii) If $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$, prove that $\sin A = \sin B = \sin C$.
- 63. Show that $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ$.
- 64. If $\alpha + \beta + \gamma = 0$, prove that $\cos \alpha + \cos \beta + \cos \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} 1.$
- 65. If in any triangle, $\tan \phi = \frac{a-b}{a+b} \cot \frac{1}{2}C$, prove that $c = (a+b) \sin \frac{1}{2}C \sec \phi$.
- 66. If $\cos \theta = \frac{a \cos \phi b}{a b \cos \phi}$, then $\frac{\tan \frac{\theta}{2}}{\sqrt{a + b}} = \frac{\tan \frac{\phi}{2}}{\sqrt{a b}}$.
- 67. If $\alpha + \beta + \gamma = \frac{1}{2}\pi$, prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 2 \sin \alpha \sin \beta \sin \gamma = 1.$
- 68. If $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$, show that one of the quantities $A \pm B \pm C$ is an odd multiple of π .
 - **69.** Show that $\sec x = \frac{2}{\sqrt{2 + \sqrt{2 + 2\cos 4x}}}$.
 - 70. If $a \sin (\theta + a) = b \sin (\theta + \beta)$, prove that $\cot \theta = \frac{a \cos a b \cos \beta}{b \sin \beta a \sin \alpha}.$
 - 71. If $\tan \beta = \frac{n \sin a \cos a}{1 n \sin^2 a}$, show that $\tan (a \beta) = (1 n) \tan a$.

In any triangle, prove that (Ex. 72 to 77):

72.
$$\frac{\cos A}{c \cos B + b \cos C} + \frac{\cos B}{a \cos C + c \cos A}$$

$$+\frac{s}{b\cos A + a\cos B} = \frac{a^2 + b^2 + c^2}{2abc}$$

73.
$$\frac{\tan \frac{A}{2}}{(a-b)(a-c)} + \frac{\tan \frac{B}{2}}{(b-c)(b-a)} + \frac{\tan \frac{C}{2}}{(c-a)(c-b)} = \frac{1}{\triangle}.$$

74.
$$\sin 3A \sin (B-C) + \sin 3B \sin (C-A) + \sin 3C \sin (A-B) = 0$$
.

75.
$$\cot B + \frac{\cos C}{\sin B \cos A} = \cot C + \frac{\cos B}{\sin C \cos A}$$

76.
$$c = (a - b) \sec \theta$$
, where $\tan \theta = \frac{2\sqrt{ab}}{a - b} \sin \frac{C}{2}$.

77.
$$a (\cos B \cos C + \cos A) = b (\cos C \cos A + \cos B)$$

= $c (\cos A \cos B + \cos C)$.

- 78. If in a triangle, $c(a+b)\cos\frac{B}{2} = b(a+c)\cos\frac{C}{2}$, show that the triangle is isosceles.
- 79. If in a triangle, a, h, c be in A.P. and the greatest angle exceeds the least by 90°, prove that

$$a:b:c=\sqrt{7}-1:\sqrt{7}:\sqrt{7}+1.$$

- 80. In the solution of triangles there can be no ambiguity except when an angle is determined by the sine or cosecant, and in no case whatever, when the triangle has one right angle; prove this.

 [Cambridge]
 - 81. If $\sin (\pi \cos \theta) = \cos (\pi \sin \theta)$, prove that

$$\cos\left(\theta\pm\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}.$$

82. If $\sin (\pi \cot \theta) = \cos (\pi \tan \theta)$, prove that either cosec 2θ or $\cot 2\theta$ is equal to $n+\frac{1}{4}$, n being an integer.

83. If a and β be the different values of θ which satisfy the equation $a \cos \theta + b \sin \theta = c$, prove that

$$\sin (a+\beta) = \frac{2ab}{a^2 + b^2}.$$

- 84. Find all the values of θ which satisfy the equation $\sin \theta + \sin 2\theta + \sin 3\theta = 1 + \cos \theta + \cos 2\theta$.
- 85. Prove that in any triangle,

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$$

- 86. If $r: R: r_1=2:5:12$, show that the triangle is right-angled.
- 87. If the angle of elevation of a cloud observed from a point at a height h above the surface of a lake be ϕ and the angle of depression of its image in the lake be θ , prove that the height of the cloud above the lake is $h \frac{\sin (\theta + \phi)}{\sin (\theta \phi)}$ assuming that the image is vertically as much below the surface as the cloud is above it.

- 88. The elevation of a tower due north of a station at A is a and at a station B due west of A is β . Prove that its altitude is $\frac{AB \sin a \sin \beta}{\sqrt{\sin^2 a \sin^2 \beta}}$ [B. H. U. I. 1934]
- 89. A man walks along a straight road and observes that the greatest angle subtended by two objects is a; from the point where this greatest angle is subtended, he walks a distance c along the road and finds that the two objects are now in a straight line which makes an angle β with the road. Prove that the distance between the objects is $c \sin a \sin \beta \sec \frac{a+\beta}{2} \sec \frac{a-\beta}{2}$. [B. H. U. I. 1936]
- 90. On the bank of a river is a column 200 ft. high supporting a statue 30 ft. high. To an observer on the opposite bank with his eye on the level of the ground the statue subtends an angle equal to that subtended by a man 6 ft. high standing at the base of the column; determine the breadth of the river.

 [B. H. U. I. 1941]

91. If O be a point inside the triangle ABC such that the angles AOB, BOC, COA are each equal to 120° and if OA = x, OB = y and OC = z, show that

$$a^{2}(y-z)+b^{2}(z-x)+c^{2}(x-y)=0.$$

92. If $\tan \theta = \frac{\tan x + t \ln y}{1 + \tan x \cdot \tan y}$, prove that

$$\sin 2\theta = \frac{\sin 2x + \sin 2y}{1 + \sin 2x \sin 2y}.$$

93. Show that

$$\frac{\sin \theta - \sin 3\theta}{\cos \theta + \cos 3\theta} + \frac{\sin \theta + \sin 3\theta}{\cos \theta - \cos 3\theta} = 2 \cot 2\theta.$$

94. Prove that in the triangle ABC

$$\frac{a^2(b^2+c^2-a^2)}{\sin 2A} = \frac{b^2(c^2+a^2-b^2)}{\sin 2B} = \frac{c^2(a^2+b^2-c^2)}{\sin 2C}.$$

95. Show that

$$\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}.$$

- 96. If $x+y=3-\cos 4\theta$, and $x-y=4\sin 2\theta$, show that $\sqrt{x}+\sqrt{y}=2$.
- 97. Show that

where $A = 2\beta + \gamma - 3\alpha$, $B = 2\gamma + \alpha - 3\beta$, $C = 2\alpha + \beta - 3\gamma$.

98. Prove that in the triangle ABC, $BC^2 \cos 2B + CA^2 \cos 2A + 2BC \cdot CA \cos (A - B) = AB^2$.

ANSWERS

Examples I. [Pages 11-14]

- 1. (i) first quadrant.
 - (iii) second quadrant.
- 2. (i) 61" 34' 44"4.
- 3. (i) '253775#.
- 4. 82° 30'; 91° 66' 6''.6; $\frac{11}{12}\pi$. 5. $\alpha: \beta = 5\pi: 24$.
- (ii) 3 3 n m.
- 6. $\frac{1}{2}\left(1-\frac{\pi}{180}\right)$ 7. 6° and 9°. 8. $\frac{1}{90}\left(D+\frac{M}{60}\right)-\frac{1}{100}\left(G+\frac{m}{100}\right)$
- 9. 1100 nearly. 10. 20° and 30°.
- 12, 20°, 40°, 80°,

20. 3.

- 13. 27°, 9°, 18°.
- 14. (i) At 28,4 min. and 48 min. past 7.

(ii) third quadrant.

(iv) fourth quadrant.

(ii) 175° 49′ 1".776.

- (ii) At 7-10. 15. 20°, 60°, 100°. 16. $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$; $\frac{\pi}{21}$, $\frac{4\pi}{21}$, $\frac{16\pi}{21}$.
- 17. 45°, 60°, 120°, 185°.

- 18. 9.
- 19. mx and nx where $x = \frac{2(10pm 9qn)}{mn(10p 9q)}$.
 - 22. 51'41 miles per hour (nearly).
- 23. 66444 miles per hour (nearly); 481445 miles (nearly).
- 24. 76'8 ft. (nearly).
- 25. 3959 miles (nearly).
- 26. 33 ft.

27. 360 vds.

21. 3 and 6.

Examples II. [Pages 24-26]

- 25. $(\sin \theta \cos \theta)^2$. 26. $\frac{1}{\tan^4 \theta} \tan^4 \theta$. 31. $\frac{a^2 b^2}{a^2 + b^4}$
- 33. $\pm \frac{\sqrt{\sec^2 a 1}}{\sec a}$; $\pm \frac{\sqrt{1 + \cot^2 \theta}}{\cot \theta}$. 34. $\frac{1}{3}$. 36. $\frac{1}{2}$. 37. 1 or $\frac{1}{2}$.
- 89. $a\frac{a^2-b^2}{2ab}$; a^2+b^2 . 43. (i) $\frac{x^2}{a^2+b^2}=1$. (ii) $xy=c^2$.

- (iii) $(bc'-b'c)^2+(ca'-c'a)^2=(ab'-a'b)^2$.
- (iv) $(a'b-b'c)(ab'-bc')=(aa'-cc')^2$.

Examples III. [Pages 35-36]

7. $\frac{\sqrt{3}}{3}$. 8. (i) 60°.

(ii) 45°. (iii) 30° (There is another

angle which is not one of the standard angles).

(iv) 45°. (v) 30°. 9. $\theta = 52\frac{1}{3}^{\circ}$, $\phi = 7\frac{1}{3}^{\circ}$.

(vi) 30°.

(vii) 80°. 10. $a = 50^{\circ}$, $\beta = 10^{\circ}$.

11. $A = 22\frac{1}{6}^{\circ}$, $B = 67\frac{1}{6}^{\circ}$, $C = 45^{\circ}$. 12. (i) $-\frac{1}{2}$. (ii) 1.

Examples IV. [Pages 49-51]

1.
$$\frac{1}{2}$$
; $-\frac{1}{\sqrt{3}}$; $\frac{2}{\sqrt{3}}$; -1. 2. $-\frac{1}{\sqrt{2}}$; $-\frac{2}{\sqrt{3}}$; $-\frac{1}{\sqrt{3}}$; $\frac{\sqrt{3}}{2}$.

4. $\frac{\sqrt{3}}{9}$. 5. (i) 1. (ii) ± 2 , $\pm \frac{2}{\sqrt{3}}$. 10. $\tan^2 \theta$; 1. 12. (i) 2.

(ii) 1. (iii) $\sin x$ or 0 according as n is odd or even. 13. $\frac{6}{6}$.

14. $\int_{0}^{\sqrt{40}}$. 15. (i) cot 26°. (ii) cos 25°. (iii) cosec 39°. (iv) cos $\frac{\pi}{9}$.

16. (i) 300°. (ii) 480°.

17. (i) 60°. (ii) 120°. 240°. (iii) 30°, 150°, 210°, 330°. (iv) 30°, 150°. (v) 30°, 185°, 150°, 315°.

Examples V. [Pages 56-59]

1. 100 J3 ft.

4. 20 √3 ft.; 20 ft. 5. 30 √2 ft.

2. 2.89... miles: 2½ miles. 3. 20 √3 ft.; 120 ft. 6. $400(\sqrt{3}+1)$ vds.

7. 40 $\sqrt{3}$ ft. 8. $\frac{1}{3}(3 \pm \sqrt{3})$ miles. 9. 22°3 miles nearly. 10. 94°64 ft. nearly. 11. 47°32 ft. nearly. 12. 60 miles per hour.

13. 50 J6 ft.

14. 40 $\sqrt{6}$ ft.; 40 $\sqrt{2}(\sqrt{7}+1)$ ft.

15. $\frac{1}{4}(\sqrt{3}+1)$ miles.

16. $5\sqrt{13}$ miles.

19. 367'38 ft.

17. 241.6... ft.: 91.6... ft. 18. 5.25... miles per hour. 20. $\frac{1}{4}\sqrt{6}(\sqrt{5}+1)$.

22. 2 miles.

28. 13.66 ft.

Examples VI. [Pages 68-70]

21. $\sin A \cos B \cos C - \sin B \cos C \cos A + \sin C \cos A \cos B$

+ sin A sin B sin C:

tan A-tan B-tan C-tan A tan B tan C 1+tan A tan B+tan C tan A-tan B tan C

22. cot A cot B cot C-cot A-cot B-cot C
cot B cot C+cot C cot A+cot A cot B-1

Examples VIII. [Pages 79-81]

Examples IX. [Pages 86-87]

16.
$$\frac{b^2-a^3}{b^2+a^2}$$
 17. (i) $2 \sin \frac{1}{2} A \sqrt{1+\sin A} + \sqrt{1-\sin A}$.

(ii) No; $2\sin\frac{1}{2}\theta = \sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$.

Examples XI. [Pages 110-111]

1.
$$n\pi \pm \frac{\pi}{4}$$
; i.e., $(2k+1)\frac{\pi}{4}$. 2. (i) $n\pi \pm \frac{\pi}{4}$. (ii) $n\pi \pm \frac{\pi}{3}$.

3.
$$2n\pi \pm \frac{\pi}{3}$$
, $(2k+1)\pi$. 4. $\frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$.

5.
$$n\pi + \frac{\pi}{4}$$
, or, $n\pi + (-1)^n \frac{\pi}{6}$. 6. $\frac{n\pi}{3}$, or, $n\pi \pm \frac{\pi}{6}$.

7.
$$\frac{r\pi}{m+(-1)^r n}$$
 8. $(2n+1)\frac{\pi}{2}$ or, $(2n+1)\frac{\pi}{4}$ or, $(2n+1)\frac{\pi}{8}$

9.
$$n\pi - \frac{\pi}{4}$$
, or, $\frac{n\pi}{2} + (-1)^n \frac{a}{2}$, where $\sin a = \frac{\sqrt{5-1}}{2}$.

11.
$$n\pi + \frac{\pi}{4}$$
 12. $(4n+1)\frac{\pi}{8}$ 13. $2n\pi + \frac{5\pi}{12}$ or, $2n\pi - \frac{\pi}{12}$

14. $(2n+1)\frac{\pi}{4}$ or, $n\pi \pm \frac{\pi}{6}$ 15. $2n\pi + \frac{\pi}{2}$ or, $2n\pi - \beta$, where β is a positive acute angle whose sine is $\frac{\pi}{2}$. 16. $\frac{1}{6}$ $\frac{1}{$

18.
$$(4n+1)_{12}^{\pi}$$
 $(n \neq 3m+2)$.

19.
$$2n\pi + \frac{1}{12}\pi$$
, or, $2n\pi + \frac{1}{12}\pi$.

20.
$$-\frac{1}{2}\pi$$
, $-\frac{1}{6}\pi$, $\frac{1}{2}\pi$, $\frac{1}{6}\pi$. 21. $\frac{1}{2}(n\pi + a)$, where $\tan a = 2$.

22.
$$2n\pi$$
. 23. $2n\pi$, $\frac{1}{6}(4n+1)\pi$. 24.

25.
$$\frac{1}{4}\pi$$
, $\frac{1}{4}\pi$. 27. (i) $\frac{1}{2}n\pi + \frac{1}{4}\pi$; $2n\pi \pm \frac{2}{3}\pi$.

(ii)
$$0, \pm \frac{\pi}{12}, \pm \frac{\pi}{6}, \pm \frac{\pi}{4}$$
 (iii) $\frac{n\pi}{3}$; $n\pi \pm \tan^{-1} \frac{1}{\sqrt{2}}$ (iv) $2n\pi - a, \frac{4n-1}{2}\pi + a$.

(v)
$$2n\pi$$
, or, $2n\pi - \frac{1}{2}\pi$. (vi) $(2n+1)\frac{\pi}{2}$, $\frac{4n+1}{14}\pi$, $\frac{4n-1}{6}\pi$.

(vii)
$$n\pi + \frac{a}{2}$$
; $(2n+1)\frac{\pi}{6} - \frac{a}{6}$ 28. $n\pi + (-1)^n 21^n 48' - 68^n 12'$.

29. (i)
$$\alpha = \beta = \frac{1}{4}\pi$$
, or, $\alpha = \frac{3}{4}\pi$, $\beta = -\frac{1}{4}\pi$.

(ii)
$$a = \frac{1}{4}\pi$$
, $\beta = \frac{1}{12}\pi$; or, $a = \frac{1}{12}\pi$, $\beta = \frac{2}{4}\pi$;

or,
$$\alpha = \frac{5}{4}\pi$$
, $\beta = \frac{5}{12}\pi$; or, $\alpha = \frac{7}{12}\pi$, $\beta = -\frac{1}{4}\pi$.

Examples XII. [Pages 119-121]

22. (i) 1. (ii) 0. (iii)
$$\frac{x+y}{1-xy}$$
. 23. $y = \frac{4x(1-x^2)}{1-6x^2+x^2}$.

24.
$$(x-y)(1+ys) = (y-z)(1+xy)$$
. 25. (i) $\frac{1}{2}$, or, -8. (ii) $\frac{a-b}{1+ab}$.

(iii)
$$\pm \frac{\sqrt{5}}{3}$$
. (iv) $\pm \frac{1}{\sqrt{8}}$. (v) $\frac{4}{3}$, or, $-\frac{2}{9}$. (vi) $\pm \frac{1}{14} \sqrt{21}$

(vii) 0, or,
$$\frac{1}{2}$$
. (viii) 0, $\pm \frac{1}{2}$. (ix) $2 - \sqrt{3}$. (x) $\frac{6 + \sqrt{6}}{8}$

Miscellaneous Examples I. [Pages 122-123]

2.
$$\pm \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}$$
 19. $a^2 + b^2 = 2(1 + c)$.

Examples XIII(a), [Pages 135-137]

1. (i) 6. (ii) -3. 2. -2. 5.
$$\frac{n}{n-1}$$
. 9. (i) 1. (ii) $1\frac{1}{2}$.

19. (i)
$$\log \frac{2}{3}$$
 i.e., '63.....

(ii)
$$4 + \frac{\log 7}{\log 8}$$
, i.e., 5.77...

(iv)
$$x = \frac{\log 3}{\log 3 - \log 2} = 2.71$$
 nearly, $y = \frac{\log 2}{\log 3 - \log 2} = 1.71$ nearly.

(v)
$$\frac{2b(2a-b)}{5ab+3ac-2b^2-bc}$$
 and $\frac{2ab}{5ab+3ac-2b^2-bc}$ where $a = \log 2$, $b = \log 3$, $c = \log 7$.

20. (i)
$$\log x = \frac{a+3b}{5}$$
; $\log y = \frac{a-2b}{5}$

Examples XIII(b). [Pages 142-144]

1. 3.2766077. 2. 1.3686646.

4. '7400827. 8. 37.6018.

6. '7928863.

5. '8455104 : 82° 17′ 21". 7. 9'8440554, 10'1559446.

8. 36° 24′ 36".

9. 58° 13′ 55″.

10. 9.6198509; 22° 36′ 28″.

12. 10·0957589. 13. 9·9147384.

14. 9.8718486.

16. $\theta = 50^{\circ} 7' 48''$ nearly.

17. '2394,

89.
$$\sqrt{\frac{y}{z} + \frac{z}{x} + \frac{x}{y}}$$
 40. 84.

Examples XIV(b). [Pages 166-168]

15. r=4; R=81.

Examples XV(a). [Pages 172-173]

- 1. 35° 5′ 49″.
- 2. 102° 1′ 28″. 3. 58° 59′ 38″.

- 4. 104° 30′: 46° 36′: 28° 54′.
- 5. (1) 88° 59′ 40.9″.

- (ii) 78° 27′ 46'86". 6. (i) 48° 11′ 23"; 58° 24′ 43"; 73° 23′ 54".
- (ii) $192^{\circ} 34' 24''$. 7. $A = 120^{\circ}$. $B = 45^{\circ}$. $C = 15^{\circ}$.
- 8. $A=45^{\circ}$, $B=30^{\circ}$, $C=105^{\circ}$, 9. $A=60^{\circ}$, $B=38^{\circ}$ 11', $C=81^{\circ}$ 49'.
- 10. $A = 105^{\circ}$, $B = 45^{\circ}$, $C = 30^{\circ}$. 11. $(\sqrt{3} + 1) : \sqrt{6} : (\sqrt{3} 1)$.
- **13.** $\sqrt{5+1}: \sqrt{5-1}$.
- 14. 3:4:5.

Examples XV(b). [Pages 176-178]

- 1. $B = 88^{\circ} 12' 48''$, $C = 21^{\circ} 47' 12''$.
- 2. $B = 56^{\circ} 19' 46'3''$, $C = 63^{\circ} 40' 13'7''$.
- 3. $A = 117^{\circ} 88' 45''$, $B = 27^{\circ} 88' 45''$,
- 4. A=94° 42′ 54", B=25° 17′ 6".
- 5. $B = 71^{\circ} 44' 29.5''$, $C = 48^{\circ} 15' 30.5''$.
- 6, (i) 70° 53′ 36"; 49° 6′ 14". (ii) 74° 13′ 50″, 35° 16′ 10″,
- (iii) $A = 64^{\circ} 21'$, $B = 77^{\circ} 25'$, c = 27.39. 7. (i) $B = 78^{\circ} 17' 39'6''$, $C = 49^{\circ} 36' 20'4''$.
 - (ii) 116° 33′ 54": 26° 33′ 54".
- 8. $A=B=75^{\circ}$, $C=30^{\circ}$, $b=2\sqrt{6}$. 9. $\sqrt{6}$, 15°, 105°.
- 10. (i) $A = 45^{\circ}$, $B = 75^{\circ}$, $c = \sqrt{6}$.
- (ii) $A = 80^{\circ}$, $B = 90^{\circ}$.

- 11. 27.0375.
- 12. 172'6436 ft.
- 13, 79°063,

- 14. (i) $A = 81^{\circ} 20'$, b = 185, c = 192.
 - (ii) b = 18.46, c = 37.16, $C = 70^{\circ} 30'$.
 - (iii) b = 118.9, c = 117.2. 16. $C=105^{\circ}$, $a=\sqrt{2}$, $c=\sqrt{3}+1$.
- 15. $C=75^{\circ}$, $a=c=2\sqrt{3}+2$. 17. 72° , 72° , 36° ; each side = $\sqrt{5+1}$.
 - 18. 8. 1.

Examples XV(c). [Pages 184-185]

- 1. (i) One solution.
- (ii) Ambiguous; two solutions.
- (iii) No solution. (iv) One solution (right-angled triangle).
- 2. (i) $C=75^{\circ}$. $A=60^{\circ}$, $a=\sqrt{6}$
 - (ii) 60°, or, 120°. or $C = 105^{\circ}$, $A = 30^{\circ}$, $a = \sqrt{2}$
- 3. $A=45^{\circ}$, $C=75^{\circ}$, $c=\sqrt{8}+1$. (no ambiguity).
- 4. Impossible.
- 8. $C = 58^{\circ} \ 56' \ 56'8'$ $A = 87^{\circ} \ 48' \ 3'7''$ or, $A = 25^{\circ} \ 41' \ 56'3''$
- 9. $B=34^{\circ}\ 27'$, $C=100^{\circ}\ 33'$.
- 10. A=5° 44' 21".
- 11. $A = 33^{\circ} 39' 34''$. $B = 86^{\circ} 20' 26''$.
- 12. $A = 80^{\circ} 86'$, $C = 64^{\circ} 14'$; or, $A = 29^{\circ} 4'$, $C = 115^{\circ} 46'$.

Miscellaneous Examples II. [Pages 186-188]

14. B=44° 25′ 39", or, 135° 34′ 21". 11. 4, 5, 6.

21. $\frac{1}{2} \{n\pi + \frac{1}{2}\pi - (a+b+c)\}$. 24. $\frac{1}{2}(n\pi + \frac{1}{2}\pi)$.

Examples XVI. [Pages 218-2147

5. $x = 38^{\circ} \cdot 10'$ nearly. 6. $\frac{1}{4\pi}$. 7. -37 nearly. 4. $\theta = \frac{1}{4}\pi$.

8. (i) x=0. (ii) $46^{\circ} 25'$ (nearly) and 90° . (iii) $22\frac{1}{2}^{\circ}$ and $112\frac{1}{2}^{\circ}$.

(iv) 3π. (v) 14° nearly. (vi) 1.19, 2.72, 4.92.

(vii) 1'16, 3'28, 4'95. (viii) ± 82. (ix) '64.

Examples XVII(a). [Pages 221-224]

14. 17° 27′ 30″. 18. 18° 26' 5'8" nearly. 4. ./2135 ft.

Examples XVII(b). [Pages 231-233]

1. $-\cos\left(\alpha + \frac{\pi}{2m}\right) / \sin\frac{\pi}{2m}$ 2. 0.

3. $\frac{\sin \{a + \frac{1}{2}(n-1)(a+\pi)\} \sin \frac{1}{2}n(a+\pi)}{\sin \frac{1}{2}(a+\pi)}$ 4. $\frac{n}{2} + \frac{\sin n\theta}{2 \sin \theta} \cos (n+1) \theta.$ 5. $\frac{1}{4} \left(\frac{3 \sin^2 na}{\sin a} - \frac{\sin^2 3na}{\sin 3a}\right)$

6. $(-1)^{n-1} \frac{\sin n\theta \sin (n+1)}{2 \cos \theta} \theta$.

7. $\frac{3}{6}n - \frac{1}{9} \frac{\sin n\pi}{\sin n} \cos (n+1) + \frac{1}{6} \cdot \frac{\sin 2n\pi}{\sin 2n} \cos 2(n+1) = 0$

8. $\cos \{\theta + \frac{1}{2}(n-1)(\theta + \frac{1}{2}\pi)\} \frac{\sin \frac{1}{2}n (\theta + \frac{1}{2}\pi)}{\sin \frac{1}{2}(\theta + \frac{1}{2}\pi)}$

9. $\frac{1}{4 \sin a} \left\{ (n+1) \sin 2a - \sin 2 (n+1) a \right\}$

10. $\frac{n}{2}\cos 2a + \frac{\cos 2(n+1)a\sin 2na}{2\sin 2a}$ 11. 1. 12. 0.

16. cosec a $\{\tan (n+1) a - \tan a\}$. 18. 0. 14. sin na.

17. cosec θ {cot θ - cot (n+1) θ }. 18. $\frac{1}{2}$ cosec a {tan (n+1) a - tan a}.

19. $\cot \theta \left\{ \cot \theta - \cot (n+1) \theta \right\} - n$. 20. $\cot \alpha - 2^n \cot 2^n \alpha$.

 $\frac{1}{2} \cdot \frac{\sin \frac{1}{2} n \theta}{\sin \frac{1}{2} \theta} \sin \frac{1}{2} (n+3) \theta - \frac{1}{4} \cdot \frac{\sin \frac{n \theta}{\theta}}{\sin \theta} \sin (n+3) \theta.$

23. $tan^{-1}\frac{n}{2+n}$. 24. $tan^{-1}\frac{n}{n+1}$. 22. $\frac{1}{2} (\tan 3^n x - \tan x)$.

25. $tan^{-1} \frac{n}{n+1}$ 26. $\frac{1}{0}$ cot $\frac{x}{0}$ -2 cot 2x.

27.
$$\frac{1}{4} \left[\frac{\sin \frac{1}{2}nx}{\sin \frac{1}{2}x} \cos \frac{1}{2} (n+3) x (1+2 \cos 2x) + \frac{\sin \frac{1}{2}nx}{\sin \frac{1}{2}x} \cos \frac{1}{2} (n+3) x \right]$$

28.
$$\frac{(n+1)\cos n\theta - n\cos (n+1)\theta - 1}{2(1-\cos \theta)}$$
 29. (a) $\frac{1}{2}n(n+1)$. (b) n^2 .

30. $\cot x \tan (n+1)x - (n+1)$; $\frac{1}{2}n(n+1)(n+2)$.

Examples XVII(c), [Pages 236-237]

1.
$$(a^2 - b^2)^2 = ab$$
. 2. a^2

1.
$$(a^2-b^2)^2=ab$$
. 2. $a^2((x+b)^2+y^2)=(x^2+y^2-b^2)^2$.

1.
$$(a^2 - b^2)^2 = ab$$
.
2. $a^2 \{(x+b)^2 + y^2\} = (x + 3y)^2 = xy^2 (x+2y)$.
4. $a^2 + b^2 = 1 + b^3 - b^4$.

5.
$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2$$
. 6. $(x^2+y^2+2ax)^2 = 4a^2(x^2+y^2)$.

7.
$$x^{\frac{3}{3}} + y^{\frac{3}{3}} = 4^{\frac{2}{3}}$$
. 8. $x^{\frac{4}{3}}y^{\frac{3}{3}} - x^{\frac{2}{3}}y^{\frac{4}{3}} = 1$. 9. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

10.
$$\frac{2x}{a} = \left(\frac{x^2}{a^2} + \frac{y^3}{b^3}\right) \left(\frac{x^3}{a^3} + \frac{y^2}{b^3} - 3\right)$$
 11. $(ax)^{\frac{3}{3}} + (by)^{\frac{3}{3}} = (a^2 - b^2)^{\frac{3}{3}}$

12.
$$\left(\frac{x}{a} + \frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{x}{a} - \frac{y}{b}\right)^{\frac{2}{3}} = 2.$$
 13. $x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = 1.$

14.
$$3a-2b=a^{3}$$
. 15. $\frac{x^{3}}{a}+\frac{y^{3}}{b}=a+b$. 16. $x^{2}+y^{3}-2\cos a=2$.

17.
$$ab = (b-a) \tan a$$
. 18. $a+b=2ab$. 19. $(ab-c)(a^2+b^2)=2ab$.

Miscellaneous Examples III [Pages 254-263]

1.
$$\frac{4}{16}\pi$$
, $\frac{1}{8}\pi$, $\frac{2}{5}\pi$. 2. 40°, 60°, 80°, $\frac{2}{9}\pi$, $\frac{1}{8}\pi$, $\frac{4}{9}\pi$. 3. 90°, 22½°, 67½°.

26.
$$\frac{(t^2+1)(t^4+1)}{t^3}$$
. 27. 1. 28. $\frac{7}{15}$. 29. $\frac{5}{17}$.

30. $x = \frac{1}{4\pi}$, $y = \frac{1}{6}\pi$. 31. (i) Possible. (ii) Impossible. (iii) Impossible. (iv) Possible.

82. (i)
$$a^2 + b^2 = 2$$
. (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$. (iii) $\left(\frac{x}{a}\right)^{\frac{3}{3}} + \left(\frac{y}{b}\right)^{\frac{3}{3}} = 1$.

46.
$$\frac{27}{535}$$
, $\frac{258}{535}$. 47. $\frac{2mn}{m^{\frac{2}{3}}-n^{\frac{2}{3}}}$. 48. $-\frac{27}{63}$. 49. $\frac{24}{25}$, $-\frac{24}{7}$, $\frac{2}{5}\sqrt{5}$.

50.
$$70 \sqrt{2}$$
. 51. $5+2 \sqrt{6}$. 52. $-\frac{2}{3}\frac{4}{5}$.

84.
$$(2n+1)^{\frac{\pi}{2}}$$
 or, $(2n+1)\pi \pm \frac{\pi}{8}$ or, $n\pi + (-1)^n \frac{\pi}{6}$ 99. 107.2 ft.

HIGHER SECONDARY EXAMINATION PAPERS

(B. O. S. E., W. B.)

1. (a) If A, B and A+B be all positive and acute, prove geometrically $\cos (A+B) = \cos A \cos B - \sin A \sin B$.

(b) Show that $\cos 6^{\circ} \cos 42^{\circ} \cos 66^{\circ} \cos 78^{\circ} = \frac{1}{16}$.

2. (a) Solve: $\tan^2 x + \cot^2 x = 2$.

(b) Solve:
$$\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$$
.

3. (a) If $\alpha + \beta = \gamma$, show that $\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos \gamma = \sin^2 \gamma.$

(b) If
$$\frac{\tan (\alpha + \beta - \gamma)}{\tan (\alpha - \beta + \gamma)} = \frac{\tan \gamma}{\tan \beta}$$
, prove that either $\sin (\beta - \gamma) = 0$
or, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$.

4. (a) If in a triangle

$$(a^2+b^2)\sin(A-B)=(a^2-b^2)\sin(A+B),$$

prove that the triangle is either isosceles or right-angled.

- (b) Find the greatest angle of a triangle whose sides are 5, 6, 7 having given $\log 6 = 7781513$, $L \cos 39^{\circ} 14' = 98890644$, diff. for 60'' = 00011032.
- 5. (a) Draw the graph of $y=\sin x+\cos x$ from x=0 to $x=2\pi$ havin given:

	10°	20°	30°	40°	50°	60°	70°	80°
sin x	•17	'34	•50	'64	.77	·87	•94	•98

(b) At each end of a horizontal line of length 2a, the angular elevation of a certain peak is θ and that at the middle point is ϕ . Prove that the vertical height of the peak is

$$\frac{a \sin \theta \sin \phi}{\sqrt{\sin (\phi + \theta)} \sin (\phi - \theta)}$$

1. (a) If A, B, A-B be positive acute angles, prove geometrically that

$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
.

(b) If
$$A+B+C=\frac{\pi}{2}$$
, prove that

$$\sin^3 A + \sin^2 B + \sin^2 C = 1 - 2 \sin A \sin B \sin C.$$

- 2. (a) Solve: $2 \sin^2 \theta + 3 \cos \theta = 0$, $(0^\circ < \theta < 360^\circ)$.
 - (b) If sec θ -cosec $\theta = \frac{4}{3}$, prove that

$$\theta = \frac{1}{2} \sin^{-1} \left(\frac{3}{4} \right).$$

3. (a) In a triangle ABC, prove that

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

(b) Two sides of a triangle are 80 and 100 feet and the included augle is 60°; find the other angles having given

4. (a) Prove that

$$\cos^2 A + \cos^2 \left(A + \frac{\pi}{3}\right) + \cos^2 \left(A - \frac{\pi}{3}\right) = \frac{3}{2}$$

(b) Prove that

$$2 \cot^{-1} 5 + \cot^{-1} 7 + 2 \cot^{-1} 8 = \frac{\pi}{4}$$

- 5. (a) Draw the graph of $\cos x \sin x$ between $x = -\pi$ and $x = \pi$.
- (b) A man standing on the bank of a river, observes that the angle subtended by a tower on the other bank just opposite to him is 60°. When he moves 60 ft. away from the bank in line with the tower he finds the angle subtended to be 30°; find the height of the tower and the breadth of the river.
- 1. (a) If C, D and C+D be all positive acute angles, prove geometrically: $\sin (C+D) = \sin C \cos D + \cos C \sin D$.

(b) If
$$A+B+C=\pi$$
 and $\sin\left(A+\frac{C}{2}\right)=n\sin\frac{C}{2}$
prove that $\tan\frac{A}{2} \cdot \tan\frac{B}{2} = \frac{n-1}{n+1}$

2. (a) Solve:— $\sin \theta + \sqrt{3} \cos \theta = \sqrt{2}$

(b) Show that, $4(\cot^{-1}3 + \csc^{-1}\sqrt{5}) = \pi$

3. (a) If
$$\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\phi}{2}$$
,

show that $\cos \phi = \frac{\cos \theta - e}{1 - e \cos \theta}$

- (b) Prove that, $\cos 36^{\circ} = \frac{1}{2}(\sqrt{5} + 1)$
- 4. (a) Show that in any triangle ABC, $a \cos A + b \cos B + c \cos C$ $= 4R \sin A \cdot \sin B \cdot \sin C$.

where R is the circumradius of the triangle,

(b) The sides of a triangle are 2, 3 and 4. Find the greatest angle of the triangle, having given:

5. (a) Draw the graph of $y=2\sin x^0+\cos x^0$, for values of x between -30 to +60, having given:

6. (b) From the top A of a building, the angles of depression of the top B and bottom C of a lamp-post BC of height h and standing on the same horizontal plane are found to be a and β . Find the height of the building and show that $AB = \frac{h \cos \beta}{\sin (\beta - a)}$

GAUHATI UNIVERSITY INTER PAPERS

- 1. (a) Write down the expressions for $\sin (A+B)$ and $\cos (A+B)$ in terms of the trigonometrical ratios of A and B; hence deduce that for $\tan (A+B)$.
 - (b) Solve: $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$ (Give general values).
 - 2. (a) If $A+B+C=\pi$, prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$,
 - (b) Show that

$$\tan^{-1} \sqrt{\frac{a(a+b+c)}{bc} + \tan^{-1}} \sqrt{\frac{b(a+b+c)}{ca}}$$

$$+\tan^{-1} \sqrt{\frac{a(a+b+c)}{ab}}.$$

8. (a) In any triangle prove that

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

- (b) The angles of a triangle, are in A. P., and the sides containing the mean angle are $\sqrt{3}+1$ and $\sqrt{3}-1$; find the other angles and the third side.
 - 4. (a) Draw a neat graph of $\cos x$ in $-\pi < x < \pi$.
 - (b) Show that

(i)
$$\tan A = \frac{\sin 2A}{1 + \cos 2A}$$
;

- (ii) $\tan A + \cot A = 2 \csc 2A$.
- 1. (a) (i) Calculate sin 221°.
 - (ii) Express 2 sin 11 θ sin θ as the difference of two terms.
- (iii) If $\tan A = \frac{4}{5}$, $\tan B = \frac{1}{5}$, find the smallest positive value of (A + B).
 - (b) If $A+B+C=\pi$, prove that

$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1.$$

2. (a) In a triangle ABC, prove that

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

If a = 8cm, b = 4cm, c = 5cm, find $\tan \frac{C}{2}$.

- (b) Solve: $\sin \theta + \sin 7\theta = \sin 4\theta$ (give general values).
- 8. (a) Express $A \cos n\theta + B \sin n\theta$ in the form $R \cos (n\theta x)$ and give the values of R and x. State the amplitude and period of

$$3\cos\theta + 4\sin\theta$$
.

(b) Show that

$$\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{2} + \tan^{-1}1 = \frac{\pi}{2}$$

- 4. Either, (a) Tabulate the values of $\sin x + \cos x$ as x varies from 0 to 2π at intervals of $\frac{\pi}{4}$.
 - Or, Find the area of $\triangle ABC$, given that a=13, b=14, c=15.
 - (b) Prove that

(i)
$$\cos A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$$

(ii)
$$\sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$$
.

PATNA UNIVERSITY QUESTIONS

- 1. (a) Show that $\cos^3 A \cos 3A + \sin^3 A \sin 3A = \cos^3 2A$.
- (b) If $x \sin^2 \theta + y \cos^2 \theta = \sin \theta \cos \theta$, and $x \sin \theta y \cos \theta = 0$, show that $x^2 + y^2 = 1$.
 - 2. (a) Establish the formula

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

- (b) Prove that $\cos^2 A + \cos^2 B + \cos^2 C 2 \cos A \cos B \cos C = 1$, if A + B = C.
 - 3. (a) Prove that in a triangle, $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$.

(b)
$$a^2 \sin (B-C) + b^2 \sin (C-A) + c^2 \sin (A-B) = 0$$
.

- 4. (a) Draw the graph of $y = \sin x + \cos x$ as x ranges from 0 to π .
- (b) Prove that $\cot A + \cot B + \cot C = \cot A \cot B \cot C$, if $A+B+C=\partial \pi$.
 - 5. (a) Prove that $\log_h n = \log_a n \times \log_h a$.
- (b) To determine the breadth AB of a canal an observer places himself at C in the straight line AB produced through C, and then walks 100 yards at right angles to the line. He then finds that AB and BC subtend angles 15° and 25° at his eyes. Find the breadth of the canal, given $L \cos 25^\circ = 9.9572757$; $L \cos 40^\circ = 9.8842540$; $L \cos 75^\circ = 9.4129962$; $\log 37279 = 4.5714648$; $\log 3728 = 3.5714759$.
 - 1. (a) Evaluate sin 18°.
 - (b) If $\sec (\phi + a) + \sec (\phi a) = 2 \sec \phi$, prove that

$$\cos\phi = \sqrt{2}\,\cos\frac{a}{2}\,\cdot$$

2. (a) If $A+B+C=\pi$, prove that

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}$$

(b) Draw the graph of $y = \tan x$ from x = 0 to $x = 2\pi$.

3. In a triangle ABC, prove that

(i)
$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$
.

- (ii) cot A, cot B, cot C are in A.P., if a2, b2, c2 are in A.P.
- 4. Two sides of a triangle are in the ratio of 9 to 7, and the included angle is 64° 12'; find the other angles, having given $\log 2 = 3010300$, $L \tan 57^{\circ}$ $54' = 10 \cdot 2025255$, $L \tan 11^{\circ}$ $16' = 9 \cdot 29993216$, $L \tan 11^{\circ}$ $17' = 9 \cdot 2999804$.
- 5. A flagstaff PN stands vertically on level ground. A base XY is measured at right angles to XN, the points X, Y, N being in the same horizontal plane, and the angle PXN and PYN are found to be α and β respectively. Prove that the height of the flagstaff is

$$\sin \alpha \sin \beta$$

 $\sqrt{\sin (\alpha - \beta)} \sin (\alpha + \beta)$ XY.

1. (a) If
$$\frac{x}{\tan (\theta + a)} = \frac{y}{\tan (\theta + \beta)} = \frac{s}{\tan (\theta + \gamma)}$$
, prove that
$$\frac{x+y}{x-y} \sin^2 (a-\beta) + \frac{y+z}{y-z} \sin^2 (\beta - \gamma) + \frac{s+x}{s-x} \sin^2 (\gamma - a) = 0.$$

- (b) Prove that $\cot A + \cot (60^{\circ} + A) + \cot (120^{\circ} + A) = 3 \cot 3A$.
- 2. (a) If $A+B+C=180^{\circ}$ and $\sin\left(A+\frac{C}{2}\right)=n\sin\frac{C}{2}$, show that

$$\tan \frac{A}{2} \tan \frac{B}{2} = \frac{n-1}{n+1}.$$

(b) If $A+B+C=180^{\circ}$, prove that

 $\sin^2 A + \sin^2 B + \sin^2 C - 2\cos A\cos B\cos C = 2.$

3. In a triangle, Prove that

(i)
$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$
.

- (ii) (b^2-c^2) cot $A+(c^2-a^2)$ cot $B+(a^2-b^2)$ cot C=0.
- 4. (a) Draw the graph of $y = \sin x$ from x = 0 to $x = \pi$, and from the graph find the angle whose sine is 7.
- (b) If a=70, b=35, $C=36^{\circ}$ 52' 12", find the other angles, having given log 3=4771213, L cot 18° 26' 6' = 10 4771213.
- 5. A flagstaff is on the top of a tower which stands on a level plane. At a certain point in the plane the tower subtends an angle α , and the flagstaff an angle β . At another points ' α ' ft. nearer the base of the tower, the flagstaff agains subtends the angle β . Show that the height

of the tower is
$$\frac{a \tan a}{1 - \tan a \tan (a + \beta)}$$

ALLAHABAD UNIVERSITY QUESTIONS

1. (a) Prove that cot
$$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{A+B}{2} \cot \frac{A-B}{2}$$
.

(b) Prove that
$$\cot \left(\frac{\pi}{4} + \theta\right) \times \cot \left(\frac{\pi}{4} - \theta\right) = 1$$
.

- 2. (a) Prove that $\frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} = \tan \theta$.
 - (b) Solve the equation $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$.

8. (a) Prove that
$$\tan \left(45^{\circ} + \frac{A}{2}\right) = \sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$$
.

(b) If
$$A+B+C=180^{\circ}$$
, prove that $\sin^{2}A + \sin^{2}B - \sin^{2}C = 2 \sin A \sin B \cos C$.

- 4. (a) Prove the formula $r = \frac{S}{s}$, where the letters have their usual meanings.
 - (b) Prove that in any triangle, $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$.
- 5. The sides of a triangle are 32, 40, and 66 feet. Find the angle opposite the greatest side, having given

 $\log 3 = 47712$, $\log 69 = 183885$, $\log 37 = 156820$,

 $\log 29 = 1.46240$, L cot 66° 10′ = 9.64517, L cot 66° 20′ = 9.64175.

6. (a) If A+B+C=2R, prove that

$$\sin (S-A) + \sin (S-B) + \sin (S-C) - \sin S$$

$$= 4 \sin \frac{A}{B} \sin \frac{B}{A} \sin \frac{C}{B}.$$

(b) In any triangle ABC, prove that

$$(b+c-a)\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=2a\cot\frac{A}{2}.$$

7. At each end of a horizontal base of length 2a it is found that the angular height of a certain peak is θ , and that at the middle point it is ϕ . Prove that the vertical height of the peak is

$$\frac{a \sin \theta \sin \phi}{\sqrt{\sin (\phi + \theta)} \sin (\phi - \theta)}$$

1. (a) Prove that

$$\frac{\cos A + \cos B}{\cos B - \cos A} = \cot \frac{A + B}{2} \cot \frac{A - B}{2}.$$

- (b) Solve the equation $4 \cos^2 \theta + \sqrt{3} = 2(\sqrt{3} + 1) \cos \theta$.
- 2. (a) If $\tan \alpha = \frac{\alpha}{6}$, and $\tan \beta = \frac{1}{17}$, prove that

$$\alpha + \beta = \frac{\pi}{4}$$

(b) Show that

$$2\cos\frac{\pi}{13} + \cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0.$$

- 3. (a) Find the value of cos 36°.
 - (b) Prove that

$$\cot A = \frac{1}{2} \left(\cot \frac{A}{2} - \tan \frac{A}{2} \right).$$

- 4. (a) Prove the formula $R = \frac{abc}{4\dot{S}}$, where the letters have their usual meanings.
 - (b) Prove that

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$$

5. In the triangle ABC, b=16, c=25 and the angle $B=33^{\circ}$ 15'. Find the remaining angles if

$$\log 2 = 30103,$$

 $L \sin 88^{\circ} 15' = 9.7390129,$

 $L \sin 58^{\circ} 56' = 9.9327616$, and

 $L \sin 58^{\circ} 57' = 9.9328876.$

6. (a) If A+B+C=2S, prove that

$$\sin (S-A) \sin (S-B) + \sin S \sin (S-C) = \sin A \sin B$$
.

- (b) If the sides of a triangle be in arithmetical progression, prove that so also are the cotangents of half the angles.
- 7. At a distance a from the foot A of a tower AB, of known height b, a flagstaff BC on the top of the tower and the tower both subtend equal argles. Find the height of the flagstaff.

1. (a) Prove that

$$\frac{\sin 7A - \sin 5A}{\cos 5A + \cos 7A} = \tan A.$$

(b) Prove that

$$\sin A = 2 \tan \frac{A}{2} / \left(1 + \tan^2 \frac{A}{2}\right).$$

- 2. (a) Prove that $\tan 2A = (\sec 2A + 1) \sqrt{\sec^2 A} 1$.
 - (b) Solve the equation $\sin \theta + \sin 3\theta + \sin 5\theta = 0$.
- 3. (a) Find the value of sin 22%.
 - (b) Prove that

$$\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}} \sin A.$$

4. (a) Prove the formula

$$r=4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

where the letters have their usual meanings.

(b) Prove that

$$r_1 r_2 r_3 = r^3 \cot^3 \frac{A}{2} \cot^2 \frac{B}{2} \cot^3 \frac{C}{2}$$

- 5. If the lengths of the greatest and least sides of a triangle be 24 and 16 feet repectively and the angle between them be 60°, find the length of the third side and the remaining angles, given $\log 2 = 30103$, $\log 3 = 4771213$, and $L \tan 19^{\circ} 6' = 9.5394217$, diff. for 1' = 4084.
 - 6. (a) If A+B+C=2S, prove that

$$\cos^2 S + \cos^2 (S - A) + \cos^2 (S - B) + \cos^2 (S - C)$$

 $=2+2\cos A\cos B\cos C$.

(b) In any triangle ABC, prove that

$$a \sin (B-C) + b \sin (C-A) + c \sin (A-B) = 0.$$

7. A tower, more than 100 feet high, consists of two parts, the lower being one-third of the whole. At a point in a horizontal plane through the foot of the tower and 40 feet from it, the upper part subtends an angle whose tangent is \(\frac{1}{2} \). Find the height of the tower.

BENARES HINDU UNIVERSITY QUESTIONS

1. Expand the determinant

$$\begin{array}{cccc} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \end{array}$$

Hence or otherwise prove that for any $\triangle ABC$,

$$a \sin (B-C) + b \sin (C-A) + c \sin (A-B) = 0.$$

2. (a) Prove that

$$2 \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$$

- (b) Solve the equation $\tan \theta = \cot \theta$.
- 3. Prove the following:

(i)
$$\frac{\sec 8A - 1}{\sec 4A - 1} = \tan 8A \cot 2A$$
.

- (ii) tan 6° tan 42° tan 66° tan 78°=1.
- (iii) $\sin (A+B) \sin (A-B) = \sin^2 A \sin^2 B$.
- 4. (a) Obtain the radius of the in-circle of a triangle in terms of the lengths of its sides.
- (b) If r and R denote respectively the radii of the inscribed and circumscribed circles of a triangle ABC, prove that

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2rR}$$

5. (a) In any triangle ABC, prove that

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

(b) In a triangle ABC, a=540 yards, b=420 yards, $\angle C=50^{\circ}$ 6'. Find the unknown angles, having given—

$$\log 2 = 30103$$

L tan 26° 3'=9'6891430

L tan 14° 20'=9'4074189

L tan 14° 21' = 9'4079453

6. The angle of elevation of a cloud from a point h feet above a lake is a and the angle of depression of its reflection in the lake is β . Prove that the height of the cloud above the lake is

$$h \frac{\sin (\beta + a)}{\sin (\beta - a)}$$

- 7. (a) Prove that $\sin^2 A + \sin^2 B \sin^2 C = 2 \sin A \sin B \cos C$, where $A + B + C = 180^\circ$.
 - (b) Solve the equation $\frac{1}{2} \cos \theta + \sin \theta = \frac{1}{2}$.
- 1. (a) If an angle subtended by an arc of length l at the centre of a circle of radius r be taken as a unit, and three angles A° , B^{g} and C radians expressed in that unit be x, y, s respectively, show that

$$x: y: z = \frac{A\pi}{18}: \frac{B\pi}{20}: 10C.$$

(b) Prove that

$$\cos (A-B) = \cos A \cos B + \sin A \sin B$$
.

- 2. Solve:
 - (i) $\tan (\pi \cot \theta) = \cot (\pi \tan \theta)$.
 - (ii) $\sin 2\theta = \cot 3\theta$.
 - (iii) $\sqrt{3} \sin \theta \cos \theta = \sqrt{2}$.
- 3. Prove that
 - (i) $\tan (\tan^{-1}x + \tan^{-1}y + \tan^{-1}z)$ = $\cot (\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$.

(ii)
$$\left(\frac{1}{r} - \frac{1}{r_1}\right) \left(\frac{1}{r} - \frac{1}{r_2}\right) \left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{16R}{r^2(a+b+c)^2}$$

where the symbols r, r_1 , r_2 , r_3 and R have their usual meanings.

4. (a) Prove that

$$16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}=1.$$

(b) In any triangle ABC, prove that $a^{2} (\cos^{2} B - \cos^{2} C) + b^{2} (\cos^{2} C - \cos^{2} A) + c^{2} (\cos^{2} A - \cos^{2} B) = 0.$

5. (a) Prove that

 $\sin 4A = 4 \sin A \cos^2 A - 4 \cos A \sin^2 A.$

(b) If the angles of a triangle are in A.P. and the lengths of the greatest and least sides be 24 and 16 feet respectively, find the length of the third side and the angles, given—

1. Prove the following:

(i)
$$\frac{1}{\sec A - \tan A} = \sec A + \tan A$$
.

- (ii) $\sin 20^{\circ} \sin 40^{\circ} \sin 60^{\circ} \sin 80^{\circ} = \frac{1}{16}$.
- (iii) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, where $A + B + C = 180^{\circ}$.
 - 2. (a) Prove that $\sin^{-1} \frac{1}{5} + \sin^{-1} \frac{1}{17} = \sin^{-1} \frac{77}{85}$.
 - (b) Solve $\tan \theta + \tan 2\theta + \sqrt{3} \tan \theta \tan 2\theta = \sqrt{3}$.
 - 3. (a) Discuss the 'ambiguous case' in the solution of triangles.
- (b) In the ambiguous case, given a, b and A, prove that the difference between the two values of c is $2\sqrt{a^2-b^2\sin^2 A}$.
 - 4. (a) In any triangle ABC, prove that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
 - (b) Prove that $r_1 + r_2 + r_3 r = 4R$.
- 5. An object is observed at three points A, B, C lying in a horizontal straight line which passes directly underneath the object. The angular elevation at B is twice that at A, and at C it is three times that at A. If AB=a, BC=b, show that the height of the object is $\frac{a}{Ob}$ $\sqrt{(a+b)(9b-a)}$.

TABLES OF LOGARITHMS, NATURAL SINES, NATURAL TANGENTS, LOGARITHMIC SINES, LOGARITHMIC TANGENTS ETC.

Table I Logarithms of numbers

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22722	17609 20413 23045 25527 27875	17898 20683 23300 25768 28103	18184 20952 28553 26007 28330	18469 21219 23805 26245 28556	18752 21484 24055 26482 28780	19033 21748 24304 26717 29003	19312 22011 24551 26951 29226	19590 22273 24797 27184 29447	19866 22531 25042 27416 29667	20140 22789 25285 27646 29885	888888	56 53 50 47	44 74 61 61	1112 99 94 89	140 132 124 117	168 158 149 141 134	196 184 174 164 156	224 210 199 188 178	252 237 223 211 201
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LOGARITHMS OF NUMBERS

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TABLE II NATURAL SINES

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0.35565 .37191 .38805 .40408	0.43575 .45140 .46690 .48226	0.51254 .52745 .54220 .55678	0.53543 .59949 .61337 .62706	0.65386 .66697 .67937 .69256 .70505	10.
0.35293 .36921 .38537 .40142	0.43313 .44680 .46433 .47971	0.51004 .52498 .53975 .55436	0.58307 .59716 .61107 .62479	0.65166 .66480 .67773 .63046	20,
0.35021 .36650 .38268 .39875	0.43051 .44620 .46175 .47716	0.50754 .52250 .53730 .55194 .56641	0.58070 .59452 .60876 .62251 .63608	0.64945 .66262 .67559 .68835	30,
0.34748 .36379 .37399 .39608 41204	0.42788 .44359 .45917 .47460	0.50508 .52002 .53484 .54951 .56401	0.57833 .59248 .60645 .62024 .63383	0.64723 .66044 .67314 .68624	4 0,
0.34475 .36108 .37730 .39341	0.42525 .44098 .45658 .47204	0.50252 .51753 .53238 .54708	0.57596 .59014 .60414 .61795	0.64501 .65825 .67129 .68412	20,
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36,	0.71121 .72337 .78531 .74708	0.76977 .78079 .79158 .80218	0.82248 .83228 .84182 .85119	0.86892 .87743 .89363
. 30,	0.71326 .72537 .73728 .74896 .76041	0.77152 .78251 .79335 .80356	0.82413 .83589 .84539 .85264 .85163	0.87036 .87883 .88701
40,	0.71529 .72737 .73924 .75088	0.77347 .78443 .79512 .80558	0.82577 .83549 .84495 .86416	0.57178 .88020 .88835 .89623
50′	0.71732 .72987 .74120 .75280	0.77531 .78622 .79688 .80730	0.82741 .83708 .84650 .85567	0.87321 .88158 .88968 .89752
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TABLE III NATURAL TANGENTS

•	0,	10,	20,	30,	,04	50,	60′			Č4	èo	Mean 3' 4' t	5' 6' 7'	ferer 6'		Z, Z	ces 4,
******	0.00000 .01746 .03492 .05241 .06993	0.00291 .02037 .03783 .05533	0.90582 .02328 .04075 .05824	0.00873 .02619 .04366 .06116	0.01164 .02910 .04658 .06408	0.01455 .03201 .04949 .06700	0.01746 .03492 .05241 .06993	888488	888888	5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5	83 88 88 88	116 116 117 117	146 146 146 146	175 175 175 175		22222	204 233 204 233 204 233 204 234 304 234
က်စ်ကိုထိရ	0.08749 .10510 .12278 .14054 .15838	0.09042 .10805 .12574 .14351	0.09335 .11099 .12869 .14648	0.09629 .11394 .13165 .14945	0.09923 .11668 .13461 .15243	0.10216 .11983 .13758 .15540	0.10510 .12278 .14054 .15838	<u> </u>	88888	59 59 60 60	88 88 89 1 89 1 90	118 118 119 120	147 147 148 149 150	176 178 178 179	~~~~~	200 200 200 200 200 200 200	06 235 06 235 07 237 08 239 09 239
ទូដូធ្លូង ទ	0.17633 .19438 .21256 .23087	0.17933 .19740 .21560 .23393	0.12233 .20042 .21864 .23700	0.18534 .20345 .22169 .24008	0.18835 .20648 .22475 .24316	0.19136 .20952 .22781 .24624 .26483	0.19438 .21256 .23087 .24933	78° 77° 75°	30 31 31 31	60 62 62 62	90 1 91 1 92 1 93 1	120 121 122 123	151 152 153 154 154	181 182 183 185 185	22222	211 212 214 214 216	1 241 2 242 4 244 6 246 7 248
\$\$7\$ \$	0.26795 .28675 .30573 .32492	0.27107 .28990 .30891 .32814	0.27419 .29305 .31210 .33136	0.27732 .29621 .31530 .33460	0.22046 .29938 .31850 .33783	0.28360 .30255 .32171 .34108	0.28675 .80573 .32492 .84433	45555	32222	63 64 65 65	94 1 95 1 96 1 98 1	125 126 129 129 131	157 158 160 162 164	188 190 194 194	555555	227 224 226 226 229	19 250 17 253 14 256 16 259 19 262

NATURAL COTANGENTS

	333 333 346	353 360 368 368 876 835	395 405 416 440	453 467 482 498 515	
265 269 273 277 2277 281 281	296 3 296 3 296 3 302 3	813 3 820 3 327 3 834 3	351 3 360 4 370 4 330 4 391 4	402 4 415 4 423 4 442 4 457 5	ού
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0.38386 .40403 .42447 .44523	0.49773 .50953 .53171 .55431	0.60086 .62187 .64941 .67451	0.72654 .75355 .78129 .50978 .83910	0.86929 .90040 .90263 1.30000	ò.
0.38053 .40065 .42105 .44175	0.48414 .50587 .52798 .55051 .57348	0.59691 .64528 .64528 .67088	0.72211 74900 77661 83498 83415	0.86419 .89515 .92709 .96008	10,
0.37720 .39727 .41763 .43828	0.49055 .50222 .52427 .54673	0.59297 .61681 .64117 .66608	0.71769 74447 77196 80020 82923	0.85912 .88992 .92170 .95451	20,
0.37388 .39391 .41421 .43481	0.47698 .49853 .52057 .54296	0.58905 .61230 .63707 .66189	0.71329 .73996 .76733 .79544	0.85408 .88473 .91633 .94896 .98270	30,
0.37057 .39055 .41081 .43136	0.47841 .49495 .51688 .53920 .56194	0.58513 .60881 .63299 .65771	0.70891 .78547 .76272 .79070	0.84906 .87955 .91099 .94345	40,
0.36727 .38721 .40741 .42791 .44872	0.46985 .49134 .51320 .53545	0.55124 .60483 .62392 .65355	0.70455 .73100 .75312 .78598 .81461	0.84407 .87441 .90569 .93797	20,
0.36397 .38336 .40403 .42447	0.46631 .48773 .50953 .53171 .55431	0.57735 .60086 .62187 .61341	0.70021 .72654 .75355 .78129	0.83910 .86929 .90040 .93252	,09
ន្តន្តន្តន្ន	88288	ส ์ส์ส์ล่ล	ลัล _{์ส} สล์	\$ \$\$ \$ \$\$	

NATURAL TANGENTS

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2, 118 118 1183 1183 1184 1180 1180 1181 1181	2' 3' 4' 118 178 237 123 184 246 127 193 194 265 138 207 276 144 216 288 150 225 300 157 235 314 164 247 329 172 259 345	2' 3' 4' 118 178 237 123 184 246 127 193 194 265 138 207 276 144 216 288 150 225 300 157 235 314 164 247 329 172 259 345	2' 3' 4' 5' 6' 118 178 237 296 355 119 18 246 307 365 1127 191 255 392 397 118 207 276 345 418 144 216 288 360 431 150 225 300 376 451 157 259 345 411 499 1172 259 345 431 517 181 272 363 453 544	Mean Differences 2' 3' 4' 6' 7' 118 178 237 296 355 414 123 184 246 307 363 436 430 127 191 255 319 382 46 397 463 132 199 265 382 397 463 138 439 463 144 216 286 360 431 503 160 451 526 157 236 314 392 471 549 576 164 247 329 411 499 576 177 269 345 431 577 603 187 549 576 545 546 546 547 549 576 547 549 576 547 549 576 547 549 548 548 548 548 548 548 548 <td< th=""></td<>
HAHAM MAMMA MA	Mean 3' 4' 178 237 184 246 199 255 199 255 207 276 207 276 225 300 225 300 225 300 225 300 225 300 225 300 227 329 247 329 259 345	Mean 3' 4' 178 237 184 246 199 255 207 276 207 276 208 288 225 390 225 345 247 329 259 345	Mean Difference 3' 4' 5' 6' 6' 6' 6' 6' 6' 6' 6' 6' 6' 6' 6' 6'	Mean Differences 3' 4' 5' 6' 7' 7' 1178 237 296 855 414 194 246 807 368 430 191 255 312 392 446 207 276 345 418 482 225 300 376 451 549 247 329 411 493 576 259 345 431 517 603 272 363 453 544 634
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2.1943 2.2998 2.4142 2.5386 2.6746	2.8239 2.9887 3.1716 3.3759 3.6059	3.8667 4.1653 4.5107 4.9152 5.3955	5.9758 6.6912 7.5958 8.7769 10.3354	12.7062 16.3499 22.9038 38.1835 114.589	30,	
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2.1609 2.2637 2.3750 2.4960 2.6279	2.7725 2.9319 3.1084 3.3052 3.5261	8.7760 4.0611 4.3897 4.7729 5.2257	5.7694 6.4348 7.2687 8.3450 9.7882	11.8262 14.9244 20.2056 31.2416 68.7501	20,	
2.2460 2.3569 2.4751 2.6051	2.7475 2.9042 3.0777 3.2709 3.4874	8.7321 4.0108 4.3315 4.7046 6.1446	5.6713 6.9138 7.1154 8.1443 9.5144			
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٥,	8.24186 8.54283 8.71880 8.84358	8.94030 9.01923 9.08589 9.14356 9.19433	9.23967 .28060 .31788 .35209	9.41300 .44034 .46594 .48998 .51264
10,	7.46373 8.30879 8.57757 8.74226 8.86128	8.95450 9.03109 9.03606 9.15245 9.2023	9.24677 .28705 .32378 .35752	9.41768 .44472 .47005 .49385
20,	7.76475 8.36678 8.60973 8.76451 8.87829	8.96825 9.04262 9.10599 9.16116 9.20999	9.25376 .29340 .32960 .36289	9.42232 .44905 .47411 .49768
30,	7.94084 8.41792 8.63968 8.78568 8.89464	8.98157 9.05386 9.11570 9.16970 9.21761	9.27,063 .29,966 .33534 .35819	9.42690 .45334 .47814 .50148
4 0,	8.06578 8.46366 8.66769 8.80585 8.91040	8.99450 9.06481 9.12519 9.17807 9.22509	9.26739 .30582 .34100 .37341 .40346	9.43143 .45758 .48218 .50523
50′	8.16268 8.50504 8.69400 8.82513 8.92561	9.00704 9.07548 9.13447 9.16628 9.23244	9.27405 .31169 .34658 .37858	9.43591 .46178 .48607 .50896
,09	8.24186 8.54282 8.71880 8.84358 8.94030	9.01923 9.08589 9.14356 9.19433	9.28060 .31788 .35209 .38368	9.44034 .46594 .48998 .51264
	සිනින්නීනී සිනින්නීනීනී	<u> </u>	73° 75° 75°	\$4%\$£
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9.54093 .56085 .57978 .59778	9.63133 .64698 .67633 .67633	9.70332 .71602 .72323 .73997	9.76218 .77268 .78280 .79256	9.81106 .81983 .83830 .93648	40,
9.54433 .56408 .58284 .60070	9.63398 .64953 .64953 .67866	9.70547 .71809 .73022 .74189	9.76395 .77439 .78445 .79415	9.81254 .82126 .83781 .83781	. 00:
9°54769 °56727 °58588 °60359 °62049	9.63662 .65205 .66682 .68098	9.70761 .72014 .73219 .74379 .75495	9.76574 .77609 .78609 .79573	9.81402 .82269 .83106 .33914 .84694	. 02
9.55102 .57044 .58889 .60646	9.63924 .65456 .65428 .68328	9-70973 -72218 -73416 -74568	9.76747 77773 78772 79731	84822 84048 84048 8410 84104 84822	10′
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LOGARITHMIC SINES

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Ç \$\$.86413 .87107 .87778	.86530 .87221 .87887	.86647 .87334 .87996	.86763 .87446 .85105	.86379 .87557 .88212	.86993 .87668 .88319	.87107 .87778 .83425	348	222	8888	38 42 82	45 43 43	58	5.23	73 75 76	88	282
2 2882	9.88425 -89050 -89653 -90235	9.88531 .89152 .99752 .90330	9.89636 .89254 .90424 .90978	9.8£741 .8£354 .8£947 .90518	3.88844 .89455 .90043 .90611	9.88948 .89554 .90139 .90704	9.89050 .89653 .90235 .90796	ક્ષેજ્રેત્રેજ્ઞજ્ઞે	0000 0000 0000	20 20 19 19	31 23 24 24	42 40 33 36	52 50 44 74 74	62 53 54	86 88 88	83 80 74 72	200 20 20 20
E & 2 & 8	9.91936 .91857 .92859 .92842	9.91425 .91942 .92441 .92921	9.91512 .92027 .92522 .92999	9.91599 .92111 .92603 .98077	9.91686 .92194 .92683 .93154	9.91772 92277 98280 98280	9.91857 .92359 .92842 .93307	88888	0.00000	17 16 16 15	25 25 23 23 23	35 32 30 30	44 42 42 42 42 42 42 42 42 42 42 42 42 4	52 50 44 47 45	61 59 55 58	65 65 65 65 66 65 67 68	78 76 73 70
82882	9.98758 94182 94598 94988	9.93826 .94252 .94660 .95052	9.93898 .94321 .94727 .95116	9.93970 .94390 .94793 .95179	9.94041 94458 94858 95242	9.94112 .94526 .94923 .95304	9.94182 .94593 .94988 .95366	88288	rrr99	13 13 13 14 15 17	120218	25 25 25 25 25	32334	443 40 38 36	50 44 48 48 48 48	55 53 50 48	64 62 64 64

LOGARITHMIC COSINES

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000000	ਚਿਚਚ ਚ	00 00 00 00 00	~~~	-H-0	, '
ង្គង្គង្គង	ಚಿಕ್ಕಗಳಿದ್ದ	8 1 1 1 1 1 1 1 1	ကိုထဲသို့ထွက်	96,440	:
9.96078 .96403 .96717 .97016	9.97567 .97821 .98284 .98494	9.98600 .98872 .99040 .99195	9.99462 .99575 .99675 .99761	9-90894 -90940 -99074 -9903 10-00000	0,
9.96017 .96349 .96665 .96966	9.97523 .97779 .98021 .98248	9.98659 .98848 .99013 .99170	9.39442 .99557 .93748 .99823	9.03882 .93934 .93991 .93900	10,
9.95960 .96294 .96614 .96917	9.97479 .97738 .97982 .98211	9.98627 .98813 .98986 .99145	9.99421 .99539 .99643 .99734	9.99876 .99926 .99964 .99958	20,
9.95902 .96240 .96562 .96868	9.97485 .97636 .97942 .98174	9.98594 .98783 .98958 .99119	9.99400 .99520 .99627 .99720	9.99866 .99919 .99959 .99986	30,
9.95844 .96185 .96509 .96818	9.97390 .97653 .97902 .98136	9.98561 .98753 .98930 .99093	9.99379 .99501 .99705 .99787	9-99856 -99911 -99953 -99987	,07
9.95786 -96129 -96456 -96767	9.97344 .97610 .97861 .98098	9.98528 .98722 .98901 .99067	9.99357 .99482 .99593 .99690	9.99845 .99903 .99947 .99978	20,
9.95728 .96073 .96403 .96717	9.97299 .97567 .97821 .98060	9.98494 .98690 .98872 .99195	999335 99462 99575 99675	9.99834 .99940 .99974 .99993	,09
88988	*****	44 444	<u> </u>	8 88388	$\lceil \rceil$

TABLE V LOGARITHMIC TANGENTS

	ಕ್ರಿಚ್ಛಬ್ಬಿಲ್ಲ	ත්ග-ා්ල්යේ	がおばばばれ	2842
ó	8.24192 8.54303 8.71940 8.81464	8.94195 9.02162 9.08914 9.14780 9.19971	9.24632 -25865 -32747 -36336	9.42805 .45750 48534 51178
10,	7.46373 8.30868 8.57788 8.74292 8.86243	8.95627 9.03361 9.09947 9.15688 9.20782	9.25865 -29535 -33865 -36909 -40212	9.43308 .46324 .48984 .51606
20,	7.76476 8.36639 8.61009 8.76525 6.87953	8-97013 9-04528 9-10956 9-16577 9-21578	9.26086 -30195 -33974 -37476	9.43806 .46694 .49430 .52031
30,	7.94086 8.41807 8.64009 8.78649 8.89598	8.98858 9.05566 9.11343 9.17450 9.22361	9.26797 .30346 .34576 .38535	9.44299 .47160 .49872 .52452
40,	8.46385 8.46385 8.66816 8.80674 8.91185	8.99662 9.06775 9.12909 9.18306 9.23130	9.27496 .31489 .35170 .38589	9.44787 .47622 .50311 .52870 .55315
50′	8.16273 8.50527 8.69453 8.82610 8.92716	$\begin{array}{c} 9.00930 \\ 9.07858 \\ 9.13854 \\ 9.19146 \\ 9.23887 \end{array}$	9.28186 .32122 .35757 .39136	9.45271 .48080 .50746 .53285
,09	8.24192 8.54308 8.71940 8.94464 8.94195	9.02162 9.08914 9.14780 9.19971 9.24632	9.28865 -32747 -36336 -39677 -42805	9.45750 .48534 .51178 .53697 .56107
i	සිනින්නිහි	88888	78,73	\$222£
	tabu Fe	98 1 57 1 78 1	71 1 65 1 60 1 56 1 52 1	49 44 42 40
Çq	iffere llatio or s	195 29 173 20 155 29	141 27 129 13 17 17 17 17 104 1	998 1 998 1 88 1 1 88 1 1 88 1
Mean 3' 4'	rences voion is in small x' or log	293 391 260 346 233 310	212 282 194 259 179 239 167 222 156 208	147 196 139 186 132 176 126 168 121 160
o Di	Differences vary so rapidly here that tabulation is impossible. For small angles of x minutes log tan x or log $\cot(30^o - x') = \log x + \frac{1}{4} \cdot 46373$.	1 488 5 433 0 388	2 354 9 323 9 299 2 278 8 261	6 245 6 232 6 220 8 210 0 201
Differences 5' 6' 7'	o rapi lible. les (30°-	586 (8 3 519 3 466	4 420 3 388 9 359 8 334 1 313	5 294 2 278 0 264 0 252
T,	$\begin{cases} \text{dily } 1 \\ \text{of } x \\ -x \end{cases}$	684 606 543	494 3 453 9 419 1 389 3 365	343 325 3325 3325 2394 1281
ào	rapidly here that ible. es of x minutes $(30^{\circ}-x)$ $= \log x + \frac{7}{4} \cdot 46373.$	782 692 621	564 518 478 445 417	392 371 352 336
6	hat ıtes 73.	879 779 698	635 582 538 500 469	442 418 396 378 362

LOGARITHMIC COTANGENTS

		LOGARITHA	HC TANGEN	STS	
347 333 322 311 302	293 284 277 271 265	260 255 251 247 244	241 235 235 234 232	230 229 223 223 225	· 6
308 296 286 277 268	260 253 246 241 241 236	231 227 223 220 220	214 212 203 208 206	205 204 203 202 202	òo
250 250 250 242 242 255	228 221 216 211 206	202 195 195 192 190	153 153 153 150 150	173 173 174 174	i-
231 222 214 208 208	195 190 175 171	173 170 167 165 165	160 158 157 157 155	154 153 152 152 152	9
193 185 179 173 173	163 158 151 151	#22.55 55.55	4348B	128 127 127 127	5,
164 148 143 138 134	130 126 123 120 118	116 113 113 110 105	55 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5	2010 1010 1010 1010 1010	٠-,-
116 1111 107 104	88 88 88 88 88 88	x 2 2 2 2 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3	311198	193329	63
77 74 72 69 67	55 62 50 50 50	3 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5	4 10 22 29 29	5225	22
35 35 35 34	88228	84446	25 25 26 26 26	25.55 25.55 25.55 25.55 25.55	-
සුඉද්ද	နှ စ္စစ္အန္	జేజీమనేటి	ಶ್ವಶ್ವಶ್ವಶ್ವಸ್ಥ	98483	
9.58418 .60641 .62785 .64858	9.68818 70717 72567 74875 76144	9.77877 .79579 .61252 .82599	9.86126 .87711 .9054 .90537	0.0000.01 5.556. 596.96. 51666.	ŗ
9.58039 .60276 .62433 .64517	9.68497 .70404 .72262 .74077	9.77591 .79297 .80975 .82620	9.85860 .87449 .89020 .90518	9.93661 .95190 .96712 .99231	10,
9.57658 .59909 .62079 .64175	9.68174 .70089 .71955 .73777	9.77303 .79015 .80697 .82352	9.85594 .87185 .88759 .90320	9.93406 .94935 .96459 .97973	50.
9.57274 .59540 .61722 .63830	9.67850 .69774 .71648 .73476	9.77015 .78732 .80419 .82078	9.85327 .86921 .88498 .90061	9-93150 -94681 -96205 -97725	30,
9.56887 .59168 .61364 .63484	9.67524 .69457 .71339 .73175	9.76725 .78448 .80140 .81803	9.85059 .86656 .88236 .89801	9.92894 .94426 .95952 .97472	,0 *
9.56498 .58794 .61004 .63135	9.67196 .69138 .71028 .72872	9.76435 .78163 .79860 .81528	9.84791 .86392 .87974 .S9541	9.92638 .94171 .95698 .97219	50′
9.56107 .58418 .60641 .62785	9.66867 .68318 .70717 .72567	9.76144 .77877 .79579 .81252	9.84528 .86126 .87711 .89281	9.92381 .93916 .95444 .96966	,09
ន្តន្តន្តន្ន	88288	ន្តង្គង្គង	ૹૹ૽ૹ૽ૹ૽	3,4884	

LOGARITHMIC TANGENTS

_	1			
ó	228 228 228 230	234 234 236 238 241	244 247 251 255 260	265 271 274 284 293
œ	202 203 203 203 203 203 203 203 203 203	208 209 212 214	217 220 223 223 221 231	241 241 253 260 263
-1 gg	771 771 771 871	180 183 183 185 186	190 192 195 202	206 211 221 221 221 221
6,0	152 152 153 153	155 156 157 158 160	162 165 167 170 173	177 181 185 190 195
Differences 5' 6' 7'	127 127 127 127 128	129 130 131 132 134	136 137 139 142 144	147 151 154 158 163
Mean 3' 4'	101 102 201	103 104 105 106 107	108 110 112 113 113	118 120 123 126 130
ર્જ	76 76 77 77	77 78 79 80	81 83 84 85	88 88 88 88 88 88
Č4	512	22 22 23 24 24 24	54 55 57 57 58	62 62 63 63 64
H	82222	28888	28882	88 8 8 8
	488 23	ૹ૾ૹ૾ૹ૽ૹ૾ૹ	ૹ૾ૹ૿ૹ૿ૹૼ	ន់នៃដំនំនំ
, ,	10.01516 .03034 .04556 .06084 .07619	10.09163 10719 12289 13874 15477	10.17101 .18748 .20421 .22123	10.25625 .27433 .29283 .31182
20	10.01263 .02781 .04302 .05829	10.08905 .10459 .12026 .13608 .15209	10'16829 '18472 '20140 '21837 '23565	10.25327 .27128 .28972 .30862
40,	10.01011 .02528 .04048 .05574	10.08647 10.1099 11764 13344 14941	10.16568 .18197 .19860 .21652	10.25081 .26825 .25661 .80643
8	10.00758 .02275 .03795 .05319	10.08390 .09959 .11562 .13079	10.16267 .17922 .19581 .22985	10.94786 .26634 .28352 .9026
20,	10.00505 .02022 .03541 .05065	10.08132 .09680 .11241 .12815 .14406	10.16016 17648 19809 20985	10.34443 .26223 .28045 .29911 .31826
10,	01870. 01870. 01870. 04810.	10.07875 .09422 .10980 .12552	10.15746 .17374 .19025 .20703	10°24148 °25923 °27738 °29596 °31503
6	10.0000 .01516 .03034 .04566	10.07619 .09163 .10719 .12289 .13874	10.15477 .17101 .18748 .20421 .22123	10°23856 °25626 °27433 °29283
	38533	22882	882388	28823

LOGARITHMIC COTANGENTS

66° 1981 8 10.98138 10.98468 10.94180 10.94460 10.9460 10.95148 10.94180 10.94180 10.94180 10.94180 10.94180 10.94180 10.9418 178 908 242 277 311 66° 19548 19682 19686	TABLE V	1	LOCARITHM	HC TANGE	NTS	
10.99138 10.39468 10.39796 10.34130 10.94465 10.74808 10.35142 235					that	٠. ا
10.99138 10.39468 10.39796 10.34130 10.94465 10.74808 10.35142 235					here	
10.99138 10.39468 10.39796 10.34130 10.94465 10.74808 10.35142 235					[4][
10.99138 10.39468 10.39796 10.34130 10.94465 10.74808 10.35142 235					rapi iole.	
10.99138 10.39468 10.39796 10.34130 10.94465 10.74808 10.35142 235					os v iposs	5
10.99138 10.39468 10.39796 10.34130 10.94465 10.74808 10.35142 235					var is in	*
10.99138 10.39468 10.39796 10.34130 10.94465 10.74808 10.35142 235					neag tion	,sc.
10.99138 10.39468 10.39796 10.34130 10.94465 10.74808 10.35142 235					flore	Ç4
10.93133 10.39463 10.38796 10.94465 10.3465 10.35142 2.95142 3.95142	35 35 37 39 39	34444	55 56 56 71	87 98	ដូច	7
10.93133 10.39463 10.89796 10.94180 10.94465 10.34603 10.39463 10.39463 35625 36170 36616 38665 39724 40091 400460 40832 41100 41562 41103 42091 40091 400460 40832 41100 415630 46182 42054 42091 400460 40832 41100 415630 46182 42054 42091 420460 10.45890 10.44685 10.44685 10.45680 10.45893 10.44685 10.44685 10.4568 10.45894 10.45892 49254 49689 50128 50570 51016 551466 51920 52378 52328 50570 51016 55146 51920 52378 56384 66341 66392 10.76870 11.06892 11.13692 11.13692 11.13692 11.13692 11.13992 11.13692 11.13992 11.39931 11	ង្គង្គង្គង	ಕ್ಕಕ್ಕಳಕ್ಕ	4 2224	ಬೆಲೆ-ತೆಹೆದೆ	46840	
10.38133 10.38463 10.38160 10.34180 10.34465 10.34805 385142 385140 385140 38516 38516 385170 38516 385170 38516 385170 38516 385170 38516 385170 38516 385170 38516 385170 38516 385170 38517	10.35142 .37215 .39359 .41582	.48822 .51466 .54250	10.60323 .63664 .67253 .71135	10.80029 .85220 .91086 .97833 11.05805	11.15536 11.28060 11.15692 11.75508 + ∞	O,
10.93133 10.39463 10.89796 10.94130 195142 35483 35692 36170 39359 39359 39724 40091 36240 41682 41061 42342 42736 10.43893 10.44682 40091 47546 47546 47546 51920 52318 55213 55731 10.57195 10.57703 10.58216 10.58216 61965 65323 66064 61411 61965 65366 65323 66064 61411 61965 65366 6536		10.45894 .48394 .51016 .53776	10.59788 .63091 .66635 .70465			10.
10.93133 10.33463 10.38162 10.54130 93142 3648 36825 36170 93124 3724 40091 38278 14.682 41961 42382 47346 10.43893 10.44288 10.44685 10.45085 46715 47130 47548 16.5426 51920 55213 55213 10.57135 10.57703 10.58216 10.58240 10.5725 66864 61441 61965 10.5765 10.57703 10.58216 10.58240 10.77363 10.76113 10.76870 10.77639 98057 93169 11.1930 11.6536 11.7380 11.19326 11.10402 11.15596 11.8081 11.38184 11.38593 11.45692 11.49473 11.38419 13.05914 + ∞	10.84465 .36516 .38636 .40832 .43113	10.45488 .47969 .50570 .53806	10.59258 .62524 .66026 .69805	10.78422 .83423 .89044 .95472 11.02987	11.12047 11.23475 11.38991 11.63311 12.23524	30,
10.38138 10.38468 38142 35488 387316 37567 39359 41961 10.43899 10.44288 46503 46716 56426 51920 56426 51920 56426 51920 56426 51920 56426 51920 56426 51920 56426 51920 56426 51920 56426 511736 11.05505 11.07284 11.15586 11.7390 11.36505 11.07284 11.15506 11.30647 11.45692 11.49473 11.45692 11.49473 11.45692 11.8737	10°34130 °36170 °38278 °40460 °42726	10.45085 .47548 .50128 .52840 .55791	10.58734 .61965 .65424 .69154 .73203	2 1		30,
10.3313 35142 37316 37316 37316 37316 46303 46303 46303 46303 46303 46303 10.75363 80039 80039 80039 811.15606 11.75806						4 0,
	10°33463 °35483 °37567 °39724 °41961	10.44288 .46715 .49254 .51920 .54729	10.57703 .60864 .64243 .67878	9		99
8 888488 28889 3444 444 44413 88488	10.38133 .35142 .37216 .39359	10.43898 .46303 .48822 .51466	10.57195 .60323 .63664 .67253	10.75363 .80029 .85220 .91086	11.05805 11.15536 11.38060 11.45692 11.75808	90,
	88788	22222 2	23,13,3	<u>*</u>	දු කුකුතුකුකු	

SOME USEFUL CONSTANTS

One radian = 57° 17' 45" nearly = 206265''; log 206265 = 5.3144.255.

 $\pi = 3.14159265...$ $\frac{1}{\pi} = 0.31830989.$ $\sqrt{2} = 1.4142135...$ $\sqrt{3} = 1.7320508...$ $\sqrt{5} = 2.2360679...$ $\sqrt{6} = 2.4494897...$ $\sqrt{7} = 2.6457513...$ $\sqrt{8} = 2.8284271...$

 $\sqrt{10} = 3.1622776...$

SOME USEFUL LOGARITHMS

 $\log 2 = 30103$ $\log 3 = 47712$ $\log 5 = 69897$ $\log 7 = 84510$